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THE BEHAVIOR OF THE SAVING RATE IN THE NEOCLASSICAL OPTIMAL GROWTH MODEL

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This paper characterizes the saving rate in the Ramsey–Cass–Koopmans model analytically with a general production function when there exist both exogenous and endogenous growth. It points out conditions involving the share of capital and the elasticities of factor and intertemporal substitution under which the saving rate path to its steady-state value exhibits overshooting or undershooting or is monotonic. Simulations illustrate these interesting dynamics. The paper also identifies the general class of production functions that render the saving rate constant along the entire transition path and hence make the Ramsey–Cass–Koopmans model isomorphic to that of Solow and Swan.

Keywords: Ramsey–Cass–Koopmans Model, Saving Rate, Elasticities of Substitution

1. INTRODUCTION

Despite its paramount importance, little is known about the behavior of the saving rate in perhaps the most popular workhorse model of macroeconomics and growth theory, the Ramsey–Cass–Koopmans optimal growth model. Only recently have there been some analytical breakthroughs on this issue. Specifically, Barro and Sala-i-Martin (2004) examine analytically the case where the production function takes the Cobb–Douglas form and show that the saving rate path to the steady state is monotonic. Smetters (2003) extends the analysis to the constant–elasticity of substitution (CES) case and proves that if the elasticity of factor substitution is different from unity, then the saving rate may exhibit nonmonotonic behavior. Finally, Gómez (2008) characterizes the global dynamics of the saving rate, also

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in the neoclassical model with CES production function, using qualitative phase diagram techniques.

This paper extends the analysis to any concave production function and derives general conditions, involving among others the elasticities of factor and intertemporal substitution, under which the saving rate manifests nonmonotonic behavior during the transition to its steady-state value. Furthermore, it contributes to the existing literature in three more ways. First, it points out cases where, when conditions are imposed on the intertemporal elasticity of substitution, the saving rate path becomes monotonic. Second, it extends the work of Kurz (1968) and Barro and Sala-i-Martin (2004) by identifying the class of production functions that render the saving rate constant along the entire transition path and hence make the Ramsey–Cass–Koopmans model isomorphic to that of Solow and Swan. Finally, it does all this not only in the case of exogenous, but also in that of endogenous unbounded growth, as in the convex model of Jones and Manuelli (1990, 2005).

The intuition behind the nonmonotonic behavior of the saving rate lies in the interaction between the income effect and the intertemporal substitution effect. As the economy develops and capital rises, its marginal product (the real interest rate) falls. This creates an income effect, which tends to raise the saving rate, and an intertemporal substitution effect, which tends to reduce it. The magnitude of each effect, and hence the overall change in the saving rate, depends on characteristics of the underlying preferences and technology, which as we show include the intertemporal elasticity of substitution and the elasticity of factor substitution.

We view the results of this paper as important for the following reasons. First of all, the resulting nonmonotonic behavior of the saving rate accords well with abundant empirical evidence.[1] Second, our results have important implications regarding the transitional dynamics of the economy and the speed of convergence. Naturally, if the elasticities of factor and intertemporal substitution alter the rate of change of the saving rate, then they will inevitably influence the speed of convergence to the steady state [see Barro and Sala-i-Martin (2004)].[2]

This paper also adds to a growing literature that tries to move away from the Cobb–Douglas restraint and toward more flexible and realistic functional forms that enhance our understanding of complex economic phenomena without masking the underlying relations. Examples of this literature that promote the use of the CES production function, besides Smetters (2003) and Gómez (2008), which are mentioned above, are Klump and de La Grandville (2000), Klump and Preissler (2000), Turnovsky (2002, 2008) Miyagiwa and Papageorgiou (2003), and Nishimura and Venditti (2004).[3] Furthermore, there have been recent works that have argued in favor of production functions that are more general than the CES. Examples include Karagiannis et al. (2005), which examines the growth implications of the variable–elasticity of substitution production function proposed by Revankar (1971); Miyagiwa and Papageorgiou (2007), which analyzes a dynamic multisector model, in which the aggregate elasticity of substitution is endogenously determined; and Papageorgiou and Saam (2008), which investigates the properties of the two-level CES aggregate production function within the Solow and Diamond growth models.

The paper is also related to the recent work of Guha (2008), who derives a set of sufficient conditions for the saving rate to be strictly increasing, strictly decreasing, or constant along the transition to the steady state.[4] In contrast, this paper points out conditions that not only ensure monotonicity in the behavior of the saving rate but also can result in the nonmonotonic (overshooting and undershooting) patterns of the saving rate that are found in the empirical literature. Moreover, it identifies the class of production functions that render the saving rate constant along the entire transition path. Finally, it derives these results not only for the case of exogenous but also for that of endogenous growth, whereas Guha works in a stationary environment where there is no growth.

The rest of the paper is organized as follows. Section 2 gives a short introduction to the model and establishes notation. Section 3 characterizes the behavior of the saving rate during the transition and points out general conditions under which the saving rate path exhibits overshooting, undershooting, or is monotonic. Section 4 analyzes the case where, within the Ramsey–Cass–Koopmans model, the saving rate remains constant over the entire transition path. Section 5 concludes the paper.

2. THE STANDARD ONE-SECTOR OPTIMAL GROWTH MODEL

The model and the notation are essentially the same as in Smetters (2003) and Barro and Sala-i-Martin (2004). The representative household maximizes[5]

$$U = \int_0^\infty \frac{c_t^{1-\theta} - 1}{1-\theta} \exp\{-(\rho - n)t\} dt, \qquad \rho,\ \theta > 0,$$

where c_t denotes consumption per capita at time t, ρ is the rate of time preference, and n is the population growth rate. The resource constraint is

$$\dot{\widehat{k}} = f(\widehat{k}_t) - \widehat{c}_t - (x + n + \delta)\widehat{k}_t, \tag{1}$$

where $\widehat{k}_t = k_t e^{-xt}$ and $\widehat{c}_t = c_t e^{-xt}$ denote capital and consumption per effective unit of labor, k_t is capital per labor unit, and $x \geq 0$ is the rate of labor-augmenting technical progress. The production function per effective unit of labor $f(\widehat{k})$ is at least twice continuously differentiable, strictly increasing, and concave $\forall \widehat{k} \in \mathbb{R}_+$ and satisfies $f(0) \geq 0$. Moreover, the production function in its extensive form $F(K, BL) = BLf(\widehat{k})$, where K, B, and L denote respectively the capital stock, the level of the technology and labor, exhibits constant returns to scale.

Applying standard techniques leads to the following equation of motion:

$$\frac{\dot{\widehat{c}}_t}{\widehat{c}_t} = \frac{f'(\widehat{k}_t) - (\delta + \rho + \theta x)}{\theta}. \tag{2}$$

The dynamics of the system (1)–(2) are well understood [see, for example, Barro and Sala-i-Martin (2004)]. In particular, if $0 \leq \lim_{\widehat{k}(t)\to\infty} f'(\widehat{k}_t) \equiv A < \rho+\delta+\theta x$, then there exists a steady-state equilibrium, which is saddle-path stable. Capital and consumption in effective units, $\widehat{k}_t$ and $\widehat{c}_t$, converge monotonically toward the steady state, so that $\dot{\widehat{k}} > 0$ and $\dot{\widehat{c}}_t > 0\, \forall t > 0$. If, on the other hand, $\lim_{\widehat{k}(t)\to\infty} f'(\widehat{k}_t) \equiv A > \rho + \delta + \theta x > 0$ then, in addition to the exogenous part generated by technical progress, there will be unbounded endogenous growth [see Jones and Manuelli (1990, 2005)]. In the latter case, if we set $x = 0$, then the exogenous part disappears; nevertheless, there will still be perpetual growth, as can be seen from equation (2).

Throughout the paper we denote the limiting value of each variable with an asterisk. If there exists a steady state in terms of effective units then $\widehat{k}^*$ and $\widehat{c}^*$ take finite values. If, on the other hand, there exists unbounded endogenous growth then $\widehat{k}^*$ and $\widehat{c}^*$ approach infinity.[6] In either case, ratios such as the gross saving rate, s_t, and the share of income consumed, $z_t = 1 - s_t$, take finite positive values, which are denoted as s^* and z^*, respectively.

Next, recall that the elasticity of substitution between capital and labor is defined as

$$\sigma(\widehat{k}_t) \equiv -\frac{f'(\widehat{k}_t)}{\widehat{k}_t f(\widehat{k}_t)} \frac{f(\widehat{k}_t) - \widehat{k}_t f'(\widehat{k}_t)}{f''(\widehat{k}_t)} > 0.^7 \tag{3}$$

When deviating from the Cobb–Douglas (CES) production function, various parameters, such as the share of capital (the elasticity of factor substitution), become endogenous variables. In addition, the Inada conditions may not be satisfied any more. In fact, the limiting properties of the production function and the behavior of the capital share are directly linked to the elasticity of factor substitution. As all these play an important role in determining the behavior of the (gross) saving rate, the following properties are very useful to our analysis.

LEMMA. *Denote the share of capital as* $f'(\widehat{k}_t)\widehat{k}_t / f(\widehat{k}_t) \equiv \alpha(\widehat{k}_t)$. *Then*

(a) $\lim_{\widehat{k}(t)\to\infty} f(\widehat{k}_t)/\widehat{k}_t = \lim_{\widehat{k}(t)\to\infty} f'(\widehat{k}_t)$;
(b) *If* $\sigma(\widehat{k}_t) \gtrless 1$ *then* $d\left[\alpha(\widehat{k}_t)\right]/d\widehat{k}_t \gtrless 0$;
(c) *If* $\lim_{\widehat{k}(t)\to 0} \sigma(\widehat{k}_t) > 1$ *then* $\lim_{\widehat{k}(t)\to 0} f(\widehat{k}_t) > 0, \quad \lim_{\widehat{k}(t)\to 0} f'(\widehat{k}_t) \leq \infty$, $\lim_{\widehat{k}(t)\to 0} \alpha(\widehat{k}_t) = 0$;
(d) *If* $\lim_{\widehat{k}(t)\to\infty} \sigma(\widehat{k}_t) > 1$ *then* $\lim_{\widehat{k}(t)\to\infty} f(\widehat{k}_t) = \infty$, $\lim_{\widehat{k}(t)\to\infty} f'(\widehat{k}_t) > 0, \quad \lim_{\widehat{k}(t)\to\infty} \alpha(\widehat{k}_t) = 1$;
(e) *If* $\lim_{\widehat{k}(t)\to 0} \sigma(\widehat{k}_t) < 1$ *then* $\lim_{\widehat{k}(t)\to 0} f(\widehat{k}_t) = 0, \quad \lim_{\widehat{k}(t)\to 0} f'(\widehat{k}_t) < \infty$, $\lim_{\widehat{k}(t)\to 0} \alpha(\widehat{k}_t) = 1$;
(f) *If* $\lim_{\widehat{k}(t)\to\infty} \sigma(\widehat{k}_t) < 1$ *then* $\lim_{\widehat{k}(t)\to\infty} f(\widehat{k}_t) < \infty$, $\lim_{\widehat{k}(t)\to\infty} f'(\widehat{k}_t) = 0, \quad \lim_{\widehat{k}(t)\to\infty} \alpha(\widehat{k}_t) = 0$.

Proof.
(a) If $\lim_{\widehat{k}(t)\to\infty} f(\widehat{k}_t) = \infty$ then apply L'Hôpital's rule. If $\lim_{\widehat{k}(t)\to\infty} f(\widehat{k}_t) < \infty$ then $\lim_{\widehat{k}(t)\to\infty} f'(\widehat{k}_t) = 0$ and the result follows.

(b) Differentiate $f'(\widehat{k}_t)\widehat{k}_t/f(\widehat{k}_t)$ with respect to $\widehat{k}_t$ and use (3).

(c–f) See Barelli and Pessôa (2003), Litina and Palivos (2008) and Palivos and Karagiannis (in press).

■

Case (d) of the above lemma warrants further comments. It is well known that a CES production function with a value of $\sigma > 1$ can lead to endogenous growth [see, for example, Solow (1956) and Pitchford (1960)]. Nevertheless, the CES is not the only production function that can deliver endogenous growth; any production function with a variable elasticity of substitution whose value is asymptotically greater than unity can result in endogenous growth. More specifically, if $\lim_{\widehat{k}(t)\to\infty} \sigma(\widehat{k}_t) > 1$ then $\lim_{\widehat{k}(t)\to\infty} f'(\widehat{k}_t) \equiv A > 0$; hence, if $A > \rho + \delta + \theta x$, then there will be unbounded endogenous growth [for details see Palivos and Karagiannis (in press)].

3. VARIABLE SAVING RATE: UNDER- AND OVERSHOOTING

It is well known that in the Ramsey–Cass–Koopmans model, along the (asymptotic) balanced growth path, the limiting values of the growth rates of consumption and capital, both measured in effective units, are equal; that is, $\gamma^*_{\widehat{c}(t)} = \gamma^*_{\widehat{k}(t)} = \gamma^*$. Using then (1) and (2), we can write γ^* and the limiting value of the (gross) saving rate $s^* = 1 - \widehat{c}^*/f(\widehat{k}^*)$ as

$$\gamma^* = \frac{f'(\widehat{k}^*) - (\delta + \rho + \theta x)}{\theta}, \tag{4}$$

$$s^* = (\gamma^* + x + n + \delta)\frac{\widehat{k}^*}{f(\widehat{k}^*)}. \tag{5}$$

Of course, if there exists only exogenous growth, then $\gamma^* = 0$.

Next let $\gamma_{z(t)}$ denote the growth rate of $z_t = 1 - s_t = \widehat{c}_t/f(\widehat{k}_t)$. The following equation is derived with the use of (1), (2), (4), and (5) and is analogous to equation (10) in Smetters (2003) and (2.95) in Barro and Sala-i-Martin (2004):

$$\begin{aligned}\gamma_{z(t)} &= \frac{\dot{z}}{z_t} = \frac{\dot{\widehat{c}}}{\widehat{c}_t} - \frac{f'(\widehat{k}_t)}{f(\widehat{k}_t)}\dot{\widehat{k}} \\ &= f'(\widehat{k}_t)\left[z_t - \frac{\theta - 1}{\theta}\right] + \frac{f(\widehat{k}^*)}{\widehat{k}^*}\left[s^*\frac{f'(\widehat{k}_t)\widehat{k}_t}{f(\widehat{k}_t)} - \frac{1}{\theta}\frac{f'(\widehat{k}^*)\widehat{k}^*}{f(\widehat{k}^*)}\right] \\ &\quad + \gamma^*\left[1 - \frac{f'(\widehat{k}_t)\widehat{k}_t}{f(\widehat{k}_t)}\right].\end{aligned} \tag{6}$$

Equation (6) applies to the case where there exists only exogenous growth as well as to the case where there exist both exogenous and endogenous growth. In particular, if there is only exogenous growth then the last term in (6) disappears because $\gamma^* = 0$. On the other hand, if $\lim_{\widehat{k}(t)\to\infty} \sigma(\widehat{k}_t) > 1$ and there is endogenous growth as well, then (6) can be simplified further because $f'(\widehat{k}^*) = f(\widehat{k}^*)/\widehat{k}^*$ [Lemma, part (a)].[8] In either case, $\gamma_z^* = 0$.

Finally, we also use the following expression, which gives the change of $\gamma_{z(t)}$ with respect to time, $\dot{\gamma}_{z(t)} \equiv d(\gamma_{z(t)})/dt$, and is obtained after differentiating (6) and using (3) and (5):

$$\dot{\gamma}_{z(t)} = f''(\widehat{k}_t)\dot{\widehat{k}}\left[z_t - \frac{\theta-1}{\theta}\right] + f'(\widehat{k}_t)\gamma_{z(t)}z_t + (x+n+\delta)\left[1-\sigma(\widehat{k}_t)\right]\frac{f''(\widehat{k}_t)\widehat{k}_t}{f(\widehat{k}_t)}\dot{\widehat{k}} \quad \textbf{(7)}$$

We are now ready to establish our first proposition. Throughout the paper, we consider the (realistic) case where a country starts with a capital stock below its steady-state value $\widehat{k}^*$.

PROPOSITION 1.

(a) *Let $\widehat{k}_1$ be a value of $\widehat{k}$ such that $\widehat{k}_1 < \widehat{k}^*$. If $\sigma(\widehat{k}_t) > 1\ \forall\, \widehat{k}_t$ and*

$$s^*\frac{f'(\widehat{k}_1)\widehat{k}_1}{f(\widehat{k}_1)} > \frac{1}{\theta}\frac{f'(\widehat{k}^*)\widehat{k}^*}{f(\widehat{k}^*)}, \quad \textbf{(1A)}$$

then the saving rate is increasing along the transition path from $\widehat{k}_1$ to $\widehat{k}^*$.

(b) *Moreover, there exists $\widehat{k}_0 < \widehat{k}_1$ such that*

$$s^*\frac{f'(\widehat{k}_0)\widehat{k}_0}{f(\widehat{k}_0)} < \frac{1}{\theta}\frac{f'(\widehat{k}^*)\widehat{k}^*}{f(\widehat{k}^*)}. \quad \textbf{(1B)}$$

In addition, the saving rate is decreasing at $\widehat{k}_0$ iff

$$1 > z_0 > 1 - \frac{1}{\theta} - \left[s^*\frac{f'(\widehat{k}_0)\widehat{k}_0}{f(\widehat{k}_0)} - \frac{1}{\theta}\right]\frac{f'(\widehat{k}^*)}{f'(\widehat{k}_0)} - \frac{\gamma^*}{f'(\widehat{k}_0)}\left[1 - \frac{f'(\widehat{k}_0)\widehat{k}_0}{f(\widehat{k}_0)}\right]. \quad \textbf{(1C)}$$

Proof.

(a) We prove this proposition for the case where there are both exogenous and endogenous growth ($\widehat{k}^* \to \infty$). The proof for the case where there is only exogenous growth and hence the economy accepts a steady state in effective units is similar. Recall from the lemma that if $\sigma(\widehat{k}_t) > 1$ and there is endogenous growth, then $f'(\widehat{k}^*)\widehat{k}^*/f(\widehat{k}^*) = 1$. Also, because $f'(\widehat{k}_1)\widehat{k}_1/f(\widehat{k}_1) < 1$, it follows from condition (1A) that $s^* > 1/\theta$ or $z^* < (\theta-1)/\theta$. Next suppose that there is a value of $t > t_{k_1}$, say t', such that $z_{t'} > (\theta-1)/\theta$, where t_{k_1} denotes the value of time that corresponds to $\widehat{k}_1$. Then (6) implies that $\gamma_{z(t')} > 0$ (because every term on the RHS is positive) and $\gamma_{z(q)} > 0\ \forall q > t'$; hence, $z^* > (\theta-1)/\theta$, which is a contradiction. Thus, $z_t < (\theta-1)/\theta\ \forall t > t_{k_1}$. Also, equation (7) implies that $\gamma_{z(t)} < 0\ \forall t > t_{k_1}$, for if $\gamma_{z(t)} \geq 0$ for some value of t, say t'', then $\dot{\gamma}_{z(q)} > 0\ \forall q > t''$, which means that $\gamma_z^* > 0$; this is also a contradiction, because along the balanced growth path s and hence z remain constant ($\gamma_z^* = 0$). But then if $\gamma_{z(t)} < 0\ \forall t > t_{k_1}$, we have that $\dot{s} > 0\ \forall t > t_{k_1}$, or that s_t is increasing along the transition path from $\widehat{k}_1$ to $\widehat{k}^*$.

(b) The proof that there exists $\widehat{k}_0$ such that (1B) holds follows immediately from the lemma [parts (c) and (d)]. If (1B) holds, then $z_t \lesseqgtr (\theta - 1)/\theta$ and $\gamma(z)$ in equation (7) can take either positive or negative values. The rest of the proof is straightforward; namely, set $\gamma_{z(0)} > 0$ in equation (6) and solve for z_0, taking into account the lemma [parts (a) and (d)].

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According to Proposition 1, the behavior of the saving rate can be nonmonotonic. More specifically, if the elasticity of factor substitution exceeds unity and condition (1A) holds, then the saving rate will be increasing before it reaches the balanced growth path and becomes constant. Furthermore, under condition (1C), there exists a value of capital $\widehat{k}_0 < \widehat{k}_1$ such that the saving rate is decreasing over the range from $\widehat{k}_0$ to $\widehat{k}_1$. Hence, looking over the entire transitional path, we see that if $\sigma(\widehat{k}_t) > 1$ and conditions (1A) and (1C) hold, then the saving rate undershoots its long-run value.

As expected, conditions (1A) and (1B) in Proposition 1 encompass the conditions that appear in Barro and Sala-i-Martin (2004) and in Smetters (2003).[9] Barro and Sala-i-Martin (2004) show that if the production function is Cobb–Douglas and $s^* = 1/\theta$ then the income and substitution effects of a change in $\widehat{k}$ offset each other. If $s^* > (<) 1/\theta$ then the income (substitution) effect dominates and the saving rate increases (decreases) during the transition. Naturally, with a general production function, what also matters is the share of capital, $f'(\widehat{k}_t)\widehat{k}_t/f(\widehat{k}_t)$, relative to its steady state value, $f'(\widehat{k}^*)\widehat{k}^*/f(\widehat{k}^*)$. That is why these terms appear in conditions (1A) and (1B). Note that if condition (1A) holds at some point $\widehat{k}_1$, then the assumption that $\sigma(\widehat{k}_t) > 1\, \forall \widehat{k}_t$ ensures that it holds everywhere between $\widehat{k}_1$ and $\widehat{k}^*$ (Lemma, part 2). Similarly, if condition (1B) holds for some value $k' < k_1$, then it holds for all values of k that are lower than k'. Finally, if $\sigma(\widehat{k}_t) = 1$ (Cobb–Douglas) and hence $f'(\widehat{k}_t)\widehat{k}_t/f(\widehat{k}_t) = \alpha$, then either (1A) is true or (1B) is true, but not both. If (1A) is true ($s^* > 1/\theta$), then only part (a) of the proposition holds and the saving rate increases during the transition. If, on the other hand, (1B) is true ($s^* < 1/\theta$), then only part (b) holds and the saving rate decreases along the entire transition path [this is the result shown in Barro and Sala-i-Martin (2004)].

We can also illustrate the results of Proposition 1 graphically (Figure 1). We do so in the $(\widehat{k}, z)$-plane because this enables us to deduce easily the behavior of the saving rate (recall that $s_t = 1 - z_t$). To save space, we analyze only the case where there is only exogenous growth $\gamma^* = 0$.[10] Setting $\dot{\widehat{k}} = 0$ [equation (1)] we obtain

$$\dot{\widehat{k}} = 0 \text{ locus:} \quad \widehat{z}_t = 1 - (x + n + \delta)\frac{\widehat{k}_t}{f(\widehat{k}_t)}, \tag{8}$$

which is downward-sloping and convex under the assumption that $\sigma(\widehat{k}_t) > 1\, \forall \widehat{k}_t$ (to make Figure 1 less cluttered, though, it is drawn as a straight line, since this does not affect our conclusions). Moreover, by setting $\gamma_{z(t)} = 0$ [equation (6)] we

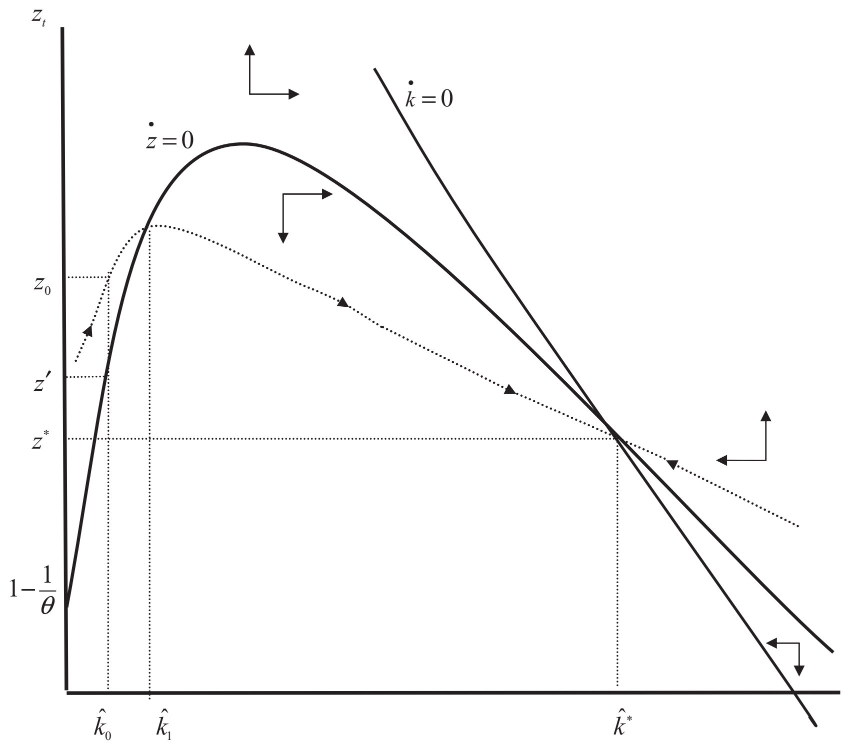

FIGURE 1. Overshooting of the consumption ratio: undershooting of the saving rate.

get

$$\dot{z} = 0 \text{ locus:} \quad \widehat{z}_t = 1 - \frac{1}{\theta} - \frac{f(\widehat{k^*})}{\widehat{k^*}} \frac{1}{f'(\widehat{k_t})} \left[s^* \frac{f'(\widehat{k_t})\widehat{k_t}}{f(\widehat{k_t})} - \frac{1}{\theta} \frac{f'(\widehat{k^*})\widehat{k^*}}{f(\widehat{k^*})} \right]. \quad \textbf{(9)}$$

The $\dot{z} = 0$ locus is more complex. The exact shape of the curve is based on the properties that appear in the lemma. We analyze the case where $\theta > 1$ and the maximum of the locus occurs at a value of $\widehat{k_t} < \widehat{k^*}$. The steady-state equilibrium $(\widehat{k^*}, z^*)$ exhibits saddle-path stability. As we see in Figure 1, for values of the initial capital stock that are greater than $\widehat{k_1}$, the consumption–output ratio z_t is monotonically decreasing and, hence, the saving rate is monotonically increasing. However, if the initial capital stock is sufficiently low, then z_t is first increasing and then decreasing; hence, z_t (s_t) exhibits overshooting (undershooting). Notice from Figure 1 that the nonmonotonicity of the $\dot{z} = 0$ locus is necessary for z to overshoot its steady-state value. Indeed, conditions 1A and 1B concern the behavior of this locus. More specifically, condition 1A is sufficient for the $\dot{z} = 0$ locus to be eventually decreasing, whereas condition 1B is necessary for the same locus to be initially increasing. The figure also provides an intuitive explanation of

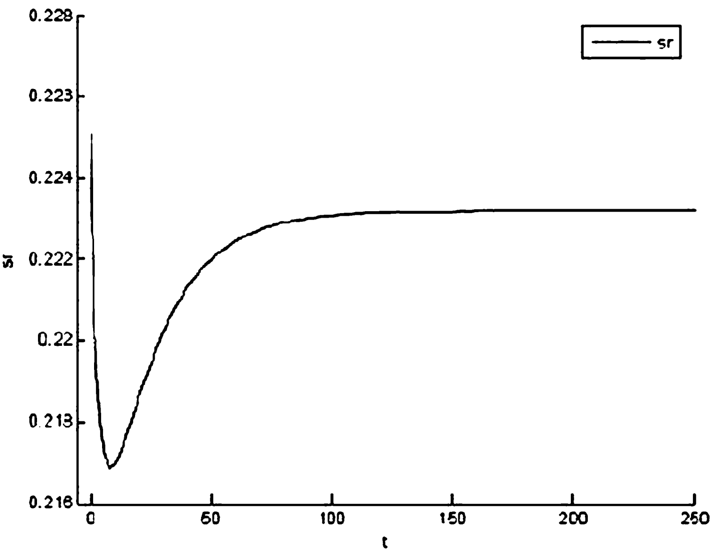

FIGURE 2. Undershooting of the saving rate: $f(\widehat{k_t}) = A(\widehat{k_t})^\alpha - \gamma$, $A = 1$, $\alpha = 0.695$, $\gamma = -61$, $n = 0.01$, $\delta = 0.0645$, $\rho = 0.035$, $\theta = 2.4$, $x = 0.01$.

condition 1C. Consider a point $(\widehat{k_0}, z_0)$ on the stable arm. Holding the capital stock the same $(\widehat{k_0})$, if we extend this point to the $\dot{z} = 0$ locus we reach another value z'. Condition 1C requires that $z_0 > z'$; that is, it requires that the consumption ratio be higher than the one that corresponds to a constant value (given by the $\dot{z} = 0$ equation). Up to point $\widehat{k_1}$, where $z_{0i} > z_i'$, as k rises so does z, whereas from point $\widehat{k_1}$ onwards, where $z_{0i} < z_i'$, as k rises z falls (z_{0i} denotes values of z on the stable arm and z_i' values on the $\dot{z} = 0$ locus).

The following example shows that undershooting can arise when there exist plausible functional forms and reasonable parameter values.

Example 1

Consider the production function $f(\widehat{k_t}) = A(\widehat{k_t})^\alpha - \gamma$, where $A > 0 > \gamma$ and $1 > \alpha > 0$. The elasticity of substitution for this production function is $1 - \alpha\gamma/[(1-\alpha)f(\widehat{k_t})]$, which is greater than unity for every finite value of k.[11] To compute the transitional dynamics of the model, we use the backward integration method proposed by Brunner and Strulik (2002), which transforms an unstable boundary value problem into a stable initial value problem by a time reversal of the dynamic system. Figure 2 presents the transitional path; clearly, the saving rate undershoots its long-run value.

COROLLARY 1. *If $\sigma(\widehat{k_t}) > 1$ $\forall \widehat{k_t}$ and θ is sufficiently high so that $z_t \leq (\theta - 1)/\theta$ $\forall t$, then the saving rate increases monotonically along the entire transition path.*

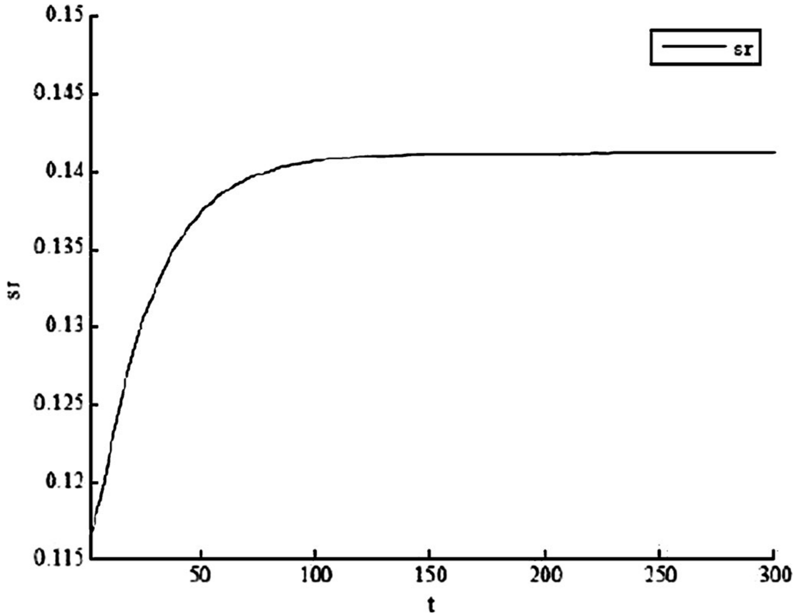

FIGURE 3. Increasing saving rate: $f(\widehat{k}_t) = A(\widehat{k}_t)^{\alpha} - \gamma$, $A = 1$, $\alpha = 0.695$, $\gamma = -61$, $n = 0.01$, $\delta = 0.0645$, $\rho = 0.035$, $\theta = 5$, $x = 0.01$.

Proof. As θ approaches infinity $z_t \leq (\theta - 1)/\theta$, $\forall t$ and not just for $t > t_{k_1}$ as in Proposition 1. Consider next equation (7). Because the first and the third terms on the RHS are both positive, it follows that $\gamma_{z(t)} < 0$, for if $\gamma_{z(t)} > 0$ then $\dot{\gamma}_{z(t)} > 0$ as well, which means that γ_z^* will never become zero. But then if $\gamma_{z(t)} < 0$, it follows that $\dot{z}_t < 0$ and $\dot{s}_t > 0\,\forall t$. ■

The intuition behind Corollary 1 is that a sufficiently high θ weakens the substitution effect and makes the income effect dominate; hence, the saving rate path is increasing.[12] The condition that $z_t \leq (\theta - 1)/\theta$ is analogous to $s^* < 1/\theta$ in the Cobb-Douglas case analyzed in Barro and Sala-i-Marin (2004). Strictly speaking, one can be certain that Corollary 1 holds always only in the limit as θ approaches infinity. Nevertheless, this Corollary gives us also a direction towards which we can search for an increasing path of the saving rate. We illustrate this point in Figure 3, where we change θ from 2.4 to 5, holding the production function and all other parameters the same as in Figure 2. The undershooting vanishes and the saving rate is strictly increasing along the entire transition path.

PROPOSITION 2.

(a) *Let* $\widehat{k}_1 < \widehat{k}^*$. *If* $\sigma(\widehat{k}_t) < 1\,\forall \widehat{k}_t$ *and*

$$s^* \frac{f'(\widehat{k}_1)\widehat{k}_1}{f(\widehat{k}_1)} < \frac{1}{\theta}\frac{f'(\widehat{k}^*)\widehat{k}^*}{f(\widehat{k}^*)} \qquad \textbf{(2A)}$$

then the saving rate is decreasing along the transition path from $\widehat{k}_1$ *to* $\widehat{k}^*$.

(b) *If there exists* $\widehat{k}_0 < \widehat{k}_1$, *such that*

$$s^* \frac{f'(\widehat{k}_0)\widehat{k}_0}{f(\widehat{k}_0)} > \frac{1}{\theta} \frac{f'(\widehat{k}^*)\widehat{k}^*}{f(\widehat{k}^*)}, \tag{2B}$$

then the saving rate is increasing at $\widehat{k}_0$ *iff*

$$0 < z_0 < 1 - \frac{1}{\theta} - \left[s^* \frac{f'(\widehat{k}_0)\widehat{k}_0}{f(\widehat{k}_0)} - \frac{1}{\theta} \frac{f'(\widehat{k}^*)\widehat{k}^*}{f(\widehat{k}^*)} \right] \frac{f(\widehat{k}^*)}{f'(\widehat{k}_0)\widehat{k}^*}. \tag{2C}$$

Proof.

(a) First note that because $\sigma(\widehat{k}_t) < 1$, $\gamma^* = 0$. Furthermore, the lemma and assumption (2A) imply that $s^* < 1/\theta$ and $z^* > (\theta - 1)/\theta$. Suppose there is a value of $t > t_{k_1}$, say t', such that $z_{t'} < (\theta - 1)/\theta$. Then (6) implies that $\gamma_{z(t')} < 0$ (because every term on the RHS is negative) and $\gamma_{z(q)} < 0\ \forall q > t'$; hence, $z^* < (\theta - 1)/\theta$, which is a contradiction. Thus, $z_t > (\theta - 1)/\theta\ \forall t > t_{k_1}$. Also, equation (7) implies that $\gamma_{z(t)} > 0\ \forall t > t_{k_1}$, for if $\gamma_{z(t)} \leq 0$ for some value of t, say t'', then $\dot{\gamma}_{z(q)} < 0\ \forall q > t''$, which means that $\gamma_z^* < 0$; this is inconsistent with the economy approaching a steady state where s and hence z are constant ($\gamma_z^* = 0$). But then if $\gamma_{z(t)} > 0\ \forall t > t_{k_1}$, we have $\dot{s} < 0\ \forall t > t_{k_1}$, or s_t is decreasing along the transition path from $\widehat{k}_1$ to $\widehat{k}^*$.

(b) The rest of the proof is straightforward.[13] Condition (2C) follows once again from equation (6); namely, set $\gamma_{z(0)} < 0$ and solve for z_0, taking into account the fact that $\gamma^* = 0$.

■

Note that according to Proposition 2, the path of the saving rate, taken as a whole, first increases and then decreases. Thus, under the specified conditions, if $\sigma(\widehat{k}_t) < 1$, then the saving rate overshoots its long-run value. Furthermore, the intuition behind conditions (2A) and (2B) is similar to that given after Proposition 1. Finally, as can easily be shown, this proposition nests the cases of Cobb–Douglas and CES.

Proposition 2 is illustrated in Figure 4. The two loci $\dot{\widehat{k}} = 0$ and $\dot{z} = 0$ are given by equations (8) and (9), respectively. We plot the case where $\theta > 1$ and the minimum of the locus occurs at a value of $\widehat{k}_t < \widehat{k}^*$ (the exact shape of each curve is again easily derived using the lemma). The steady-state equilibrium is the point $(\widehat{k}^*, z^*)$, which is saddle-path stable. For values of the initial capital stock that are greater than $\widehat{k}_1$, the consumption ratio z_t is monotonically increasing and, hence, the saving rate is monotonically decreasing. However, if the initial capital stock is sufficiently low, then z_t is first decreasing and then increasing; hence, z_t exhibits undershooting. The saving rate behaves exactly the opposite and, therefore, it exhibits overshooting. Just as in Proposition 1, the conditions that appear in Proposition 2 concern the behavior of the $\dot{z} = 0$ locus. Notice that for undershooting of z to occur this locus must be nonmonotonic. Condition 2A is sufficient for it to be eventually increasing, whereas condition 2B is necessary for it to be initially decreasing. Finally, condition 2C follows the same way as condition 1C in Proposition 1 and Figure 1. Consider first a point $(\widehat{k}_0, z_0)$ on the stable arm. Holding the capital stock the same $(\widehat{k}_0)$, if we extend this point to the

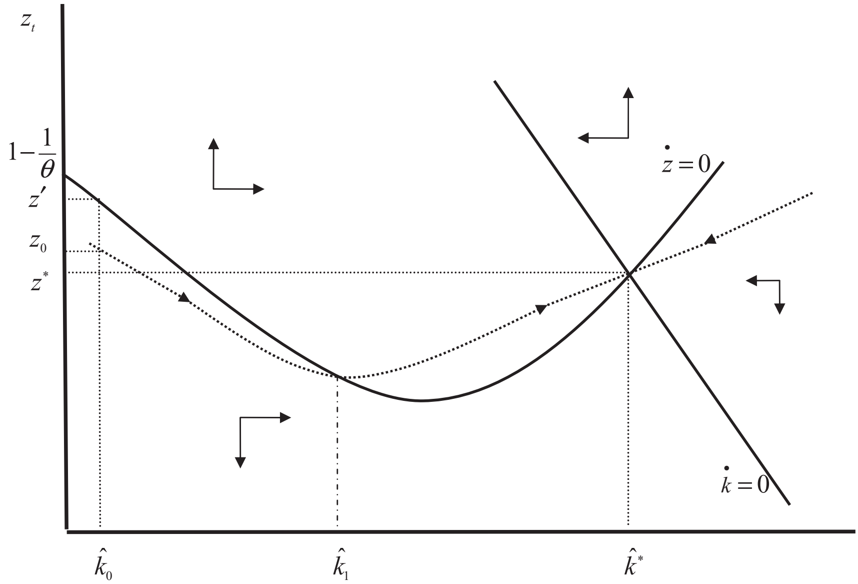

FIGURE 4. Undershooting of the consumption ratio: overshooting of the saving rate.

$\dot{z} = 0$ locus we reach another value z'. Condition 2C requires that $z_0 < z'$; that is, it requires that the consumption ratio be lower than the one that corresponds to a constant value (given by the $\dot{z} = 0$ equation). Thus, up to point $\widehat{k}_1$, as k rises, z falls.

Example 2

Consider the production function $f(\widehat{k_t}) = 1 - e^{-b\widehat{k_t}}$, $b > 0$. The elasticity of substitution for this function is $(1/b\widehat{k_t}) - [e^{-b\widehat{k_t}}/f(\widehat{k_t})]$, which is less than unity for every value of k. Using the same numerical method as in Example 1 we can compute the transitional dynamics of the economy. The path of the saving rate, which is depicted in Figure 5, shows that in this case s overshoots its long-run value.

Notice in Proposition 2 that given condition (2B) and the fact that $z_t > 0$, a necessary condition for (2C) to hold is $\theta > 1$. In other words, if $\theta \leq 1$, then condition (2C) in Proposition 2 cannot hold and, thus, the saving rate path will be monotonic. More formally, consider the following corollary.

COROLLARY 2. $\sigma(\widehat{k_t}) \leq 1\, \forall \widehat{k_t}$ *and* $\theta \leq 1$ *then the saving rate declines monotonically along the entire transition path.*[14]

Proof. First, note that because $\sigma(\widehat{k_t}) \leq 1$, $\gamma^* = 0$. Furthermore, if $\theta \leq 1$ then $s_t < 1 \leq 1/\theta$ and $z_t > 0 \geq (\theta-1)/\theta\ \forall t$ and not just for $t > t_{k_1}$, as in Proposition 2. Consider next equation (7). Because the first and third terms on the right-hand

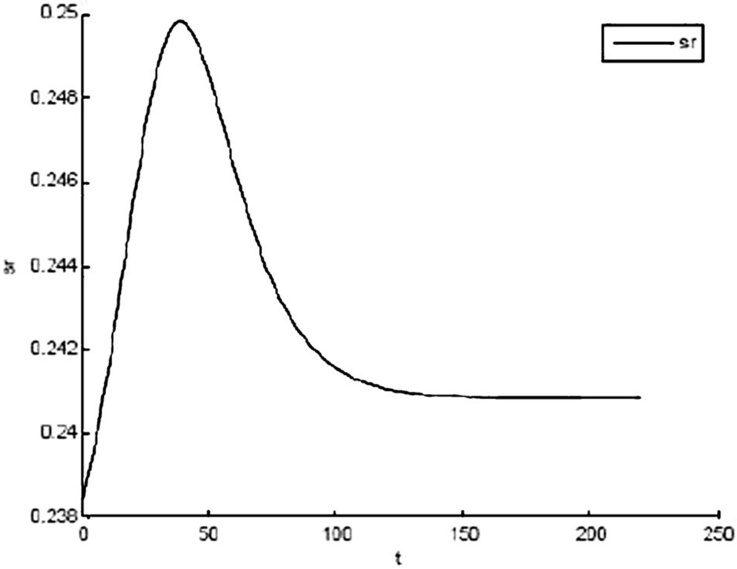

FIGURE 5. Overshooting of the saving rate: $f(\widehat{k_t}) = 1 - e^{-b\widehat{k_t}}$, $b = 0.64$, $n = 0.01$, $\delta = 0.0645$, $\rho = 0.028$, $\theta = 7.3$, $x = 0.01$.

side are both negative, it follows that $\gamma_{z(t)} > 0$, for if $\gamma_{z(t)} < 0$ then $\dot{\gamma}_{z(t)} < 0$ as well, which means that γ_z^* will never become zero. But then if $\gamma_{z(t)} > 0$, it follows that $\dot{z}_t > 0$ and $\dot{s}_t < 0\,\forall t$, that is, the saving rate declines monotonically over time along the entire transition path. ■

Example 3

Consider the case where the production function is Cobb–Douglas $f(\widehat{k_t}) = k^\alpha$, $\alpha < 1$, and the share of capital equals the inverse of the intertemporal elasticity of substitution, that is, $\alpha = \theta$. It is known that in this case the model can be solved analytically and the saving rate is [see Smith (2006, equation 20)]

$$s_t = 1 - \frac{\rho + (1-\alpha)\delta - \alpha n}{\alpha}\left[(\widehat{k^*})^{1-\alpha} + \left[(\widehat{k_0})^{1-\alpha} - (\widehat{k^*})^{1-\alpha}\right] e^{-(1-\alpha)\frac{\delta+\rho+\theta x}{\alpha}t}\right].$$

If $\widehat{k_0} < \widehat{k^*}$, then $ds_t/dt < 0$. That is, the saving rate decreases monotonically over the transition to its steady-state value; the substitution effect dominates the income effect along the entire transition path.

4. CONSTANT SAVING RATE

As mentioned above, when the production function is Cobb–Douglas and $s^* = 1/\theta$, the saving rate is constant over the entire transition [see Kurz (1968) and Barro and Sala-i-Martin (2004)].[15] In other words, this is a case where the income and substitution effects cancel each other. The following proposition generalizes this result. It identifies the general class of production functions that yield a constant saving rate and hence make the Ramsey–Cass–Koopmans and the Solow–Swan models isomorphic.

PROPOSITION 3. *The saving rate is constant over the entire transition if and only if the inverse of the production function* $\widehat{k}(f)$ *belongs to the class*

$$\widehat{k}(f) = \begin{cases} f^b\left(A - \dfrac{\theta s^* - 1}{\delta + \rho + \theta x}\dfrac{f^{-b+1}}{-b+1}\right), & b \neq 1 \\ f\left(A - \dfrac{\theta s^* - 1}{\delta + \rho + \theta x}\ln f\right), & b = 1 \end{cases}, \tag{10}$$

where $b \equiv \theta(x+n+\delta)/(\delta+\rho+\theta x)$ *and A is a constant of integration.*

Proof.

($\Rightarrow$) If the saving rate is constant, $s_t = s^* \, \forall t$, then $\gamma_{z(t)} = 0 \, \forall t$. Setting equation (6) equal to zero and using equations (4) and (5) yields

$$s^* = \frac{1}{\theta} + (x+n+\delta)\frac{k_t}{f(k_t)} - \frac{\delta+\rho+\theta x}{\theta}\frac{1}{f'(k_t)} \tag{11}$$

or

$$\frac{dk}{df} - \frac{\theta(x+n+\delta)}{\delta+\rho+\theta x}\frac{k}{f} = \frac{1-\theta s^*}{\delta+\rho+\theta x},$$

which is a first-order differential equation. The solution of this equation yields (10).

($\Leftarrow$) If the production function is (10) then (11) holds. When (4), (5), and (11) are used (6) becomes

$$\gamma_{z(t)} = f'(\widehat{k}_t)[s^* - s_t].$$

Suppose $s^* > s_t$ for some $t = t'$, then $\gamma_{z(t)} > 0$ and $\gamma_{s(t)} < 0$ for all $t > t'$, a result that is inconsistent with s approaching its steady-state value s^*. Similarly, $s^* < s_t$ for any $t = t'$ can be ruled out because it implies that $\gamma_{z(t)} < 0$ and $\gamma_{s(t)} > 0$ for all $t > t'$, a result that is again inconsistent with s approaching its steady-state value. Thus, $s_t = s^* \, \forall t$. ■

Of course, in general, more restrictions on the parameter values might be needed to ensure that the inverse of (10) exists and that it is a proper production function. We can illustrate the dynamics of this economy as well. In Figure 6 we consider the case where $\gamma^* = 0$. The locus $\dot{\widehat{k}} = 0$ is the same as before [equation (8)]. Also, the $\dot{z} = 0$ locus is given by equation (9), which, after equations (4) and (5) are used, becomes

$$\widehat{z}_t = 1 - \frac{1}{\theta} - (x+n+\delta)\frac{\widehat{k}_t}{f(\widehat{k}_t)} + \frac{1}{\theta}\frac{\delta+\rho+\theta x}{f'(\widehat{k})}. \tag{12}$$

Moreover, it follows from equation (11) that the term

$$(x+n+\delta)\frac{k_t}{f(k_t)} - \frac{\delta+\rho+\theta x}{\theta}\frac{1}{f'(k_t)}$$

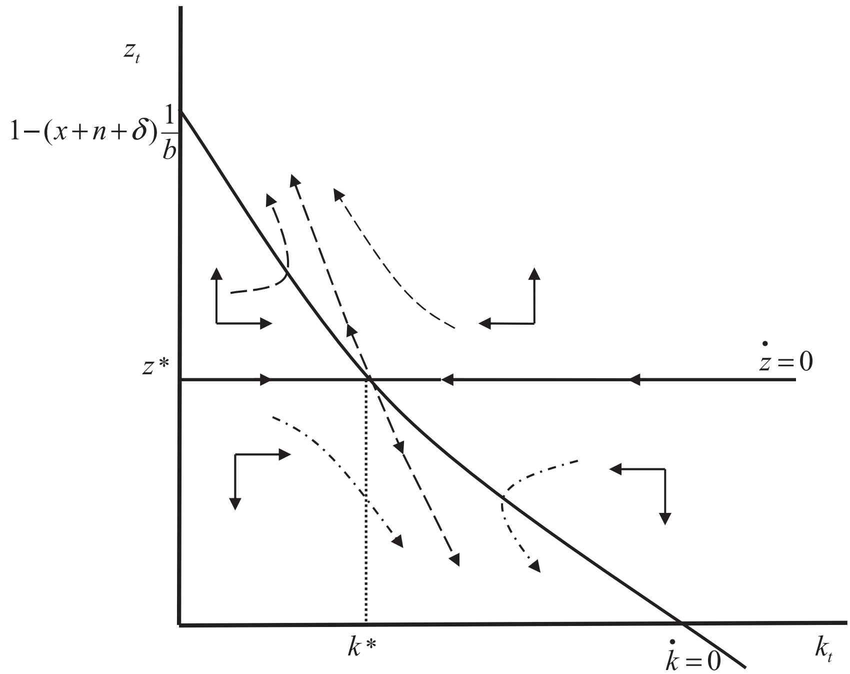

FIGURE 6. Constant saving rate.

is independent of time and, hence, the $\dot{z} = 0$ locus becomes parallel to the horizontal axis.

Given that z_t is independent of time, we can write equation (1) as

$$\dot{\widehat{k_t}} = (1 - z^*) f(k) - (x + n + \delta)k,$$

which is the fundamental Solow equation. As shown in Proposition 4 below, however, there is still an essential difference between the Ramsey model with a (10) production function and the Solow model; namely, the saving rate in the Ramsey model is related to the underlying parameters and cannot be chosen arbitrarily.

Example 4

Note that if $s^* = 1/\theta$ then (10) yields $f(\widehat{k}) = A^{-1/b}(\widehat{k})^{1/b}$. Of course, one needs to assume that $A > 0$ for positive values of output and $b \geq 1$ for concavity. If $b > 1$ then we obtain the Cobb–Douglas production function; this is the result found in Kurz (1968) and Barro and Sala-i-Martin (2004). If $b = 1$, then we have the so-called "*Ak*" production function.

Example 5

Consider next the case where $b = 2$. Then (10) becomes

$$Af^2 + \frac{\theta s^* - 1}{\delta + \rho + \theta x} f - \widehat{k} = 0,$$

which yields

$$f(\widehat{k}) = \frac{-\dfrac{\theta s^* - 1}{\delta + \rho + \theta x} + \left[\left(\dfrac{\theta s^* - 1}{\delta + \rho + \theta x}\right)^2 + 4A\widehat{k}\right]^{1/2}}{2A} \quad \textbf{(13)}$$

(the other root is rejected because it yields negative values for output). We assume that $A > 0$ so that output takes positive values for all possible parameter values. Without further restrictions, the function (13) is a proper production function whose properties depend on the values of its parameters. For example, if $\theta s^* < 1$, then $\lim_{\widehat{k}\to 0} f(\widehat{k}) > 0$, $\lim_{\widehat{k}\to\infty} f(\widehat{k}) = \infty$, $f'(\widehat{k}) > 0 > f''(\widehat{k})$, $\lim_{\widehat{k}\to 0} f'(\widehat{k}) > 0$, $\lim_{\widehat{k}\to\infty} f'(\widehat{k}) = 0 = \lim_{\widehat{k}\to 0} \widehat{k} f'(\widehat{k})/f(\widehat{k})$, $\lim_{\widehat{k}\to\infty} \widehat{k} f'(\widehat{k})/f(\widehat{k}) = 1/2$, $\lim_{\widehat{k}\to 0} \sigma(\widehat{k}) = \infty$, and $\lim_{\widehat{k}\to\infty} \sigma(\widehat{k}) = 1$.

Just as in the case of Section 2, there is a close connection between the elasticity of substitution, the share of capital and the saving rate. Consider the following proposition.

PROPOSITION 4. *The saving rate is constant if and only if*

$$\sigma(\widehat{k}_t)\alpha(\widehat{k}_t) = 1/b \, \forall \widehat{k}_t, \quad \textbf{(14)}$$

where it may be recalled that $b \equiv \theta(x + n + \delta)/(\delta + \rho + \theta x)$.

Proof. Notice from (11) that the saving rate is constant if and only if

$$(x + n + \delta)\frac{k_t}{f(k_t)} - \frac{\delta + \rho + \theta x}{\theta}\frac{1}{f'(k_t)}$$

is independent of time. Upon differentiation we see that this is the case if and only if (14) holds. ■

Proposition 4 determines when the income and intertemporal substitution effects offset each other. Using equation (7), we see that in the case of exogenous growth, that is, $\gamma^* = 0$, $s^* = (1/\sigma^*\theta)$ (once again in the case of Cobb–Douglas $\sigma = 1$ and we obtain the familiar result that if $s^* = 1/\theta$ then s_t is constant). In the endogenous growth case, on the other hand, where $\lim_{\widehat{k}\to\infty} \alpha(\widehat{k}_t) = 1$, we have $\sigma^* = 1/b$ and $s(t) = s^* = (\gamma^* + x + n + \delta)/(\theta\gamma^* + \delta + \rho + \theta x)$.

5. CONCLUSIONS

This paper has characterized analytically the behavior of the saving rate along the transition path for the Ramsey–Cass–Koopmans model for any concave production function and for cases of both endogenous and exogenous growth. It has shown that for an elasticity of factor substitution greater (less) than unity, the saving rate path may exhibit undershooting (overshooting). Nevertheless, by imposing conditions on the elasticity of intertemporal substitution, we can ensure monotonicity. Finally, the paper identifies the general class of production functions that render the saving rate constant over the entire transition path, as in the Solow–Swan model, with the crucial difference, however, that in this model the saving rate is endogenously determined.

As mentioned in the Introduction, there is a growing literature that finds evidence against the Cobb–Douglas specification and in favor of more flexible functional forms. In view of this, the results of this paper are important and may lead to a better understanding of the different saving patterns that are observed in the data.

NOTES

1. Maddison (1992) examines several countries and finds that after World War II the saving rate exhibits overshooting in most of them. Overshooting of the saving rate is also found in, among others, Bosworth et al. (1991) for the United States, Canada, and Japan; Christiano (1989) for Japan; and Chari et al. (1996) for South Korea. Tease et al. (1991) and Loayaza et al. (2000) report similar trends across the world.

2. Turnovsky (2002, 2008) analyzes how a change in these elasticities of substitution affects the speed of convergence in the neoclassical growth model with CES production function.

3. The rejection of the Cobb–Douglas production function and the use of more flexible functional forms has recently found empirical support in, among others, the papers of Duffy and Papageorgiou (2000), Antràs (2004), Masanjala and Papageorgiou (2004), Karagiannis et al. (2005), and Klump et al. (2007).

4. We are grateful to a referee for calling our attention to the work of Guha (2008).

5. We assume that the utility function takes the constant–intertemporal elasticity of substitution form because this is a necessary condition for the existence of a balanced growth path; see King et al. (1988) and Palivos et al. (1997).

6. Hence, expressions such as $f'(\widehat{k}^*)$ should be understood as $\lim_{\widehat{k}\to\widehat{k}^*} f'(\widehat{k}_t)$ if there is exogenous growth only and as $\lim_{\widehat{k}\to\infty} f'(\widehat{k}_t)$ if there is endogenous growth. We abuse the notation somewhat in order to make the two cases of just exogenous and of exogenous as well as endogenous growth immediately comparable.

7. We remind the reader that for a general production function $F(K, L)$ the elasticity of substitution is defined as $d\ln(K/L)/d\ln(F_L/F_K)$. Under constant returns to scale and with two factors of production this can be written as $F_K F_L/F F_{KL}$, which, in terms of the intensive form f, is equal to the expression given in (3).

8. Recall that if there is endogenous growth $\widehat{k}^* \to \infty$.

9. For example, if the production function is Cobb–Douglas then $f'(\widehat{k}_t)\widehat{k}_t/f(\widehat{k}_t) = f'(\widehat{k}^*)\widehat{k}^*/f(\widehat{k}^*) = \alpha$ and (1A) simplifies to $s^* < 1/\theta$, as in Barro and Sala-i-Martin. Also, if the production function is CES then condition (1A) simplifies to $s^*[f'(\widehat{k}_1)/f'(\widehat{k}^*)]^{1-\sigma} > 1/\theta$ as in Smetters (2003), Proposition 1B.

10. The case of endogenous growth involves a similar analysis after the system is transformed into one that accepts a steady state.

11. Note, however, that $\lim_{k\to\infty} f'(\widehat{k}) = 0$ and hence this production function cannot yield endogenous unbounded growth.

12. Most empirical estimates of the intertemporal elasticity of substitution are quite small, certainly less than one [for example, see Hall (1988), Ogaki and Reinhart (1998), and Biederman and Goenner (2008)]; hence $\theta > 1$.

13. Notice that, in contrast to Proposition 1, the existence of $\widehat{k}_0$ such that condition (B) in Proposition 2 holds is not guaranteed any more. The same is true in Smetters (2003). We note that there is a small typo in Smetters (2003) toward the end of the proof of Proposition 1, case (A), on p. 705; namely, the paper appears to claim that if $f(\widehat{k}) = [\alpha\widehat{k}^{1-1/\sigma_{KL}} + (1-\alpha)t]^{1/(1-1/\sigma_{KL})}$ and $\sigma_{KL} < 1$, then $\lim_{\widehat{k}\to 0} f'(\widehat{k}) = \infty$. This should read, instead, $\lim_{\widehat{k}\to 0} f'(\widehat{k}) = \alpha^{1/(1-1/\sigma_{KL})} < \infty$.

14. The case where $\sigma = 1$ is also shown in Barro and Sala-i-Martin (2004).

15. Kurz (1968) reaches this conclusion by analyzing the inverse optimal problem for the Solow–Swan model; that is, given a consumption path, he determines a class of objective functionals that would optimally imply such a path.

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