Education and Growth: A Simple Model with Complicated Dynamics*

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Abstract

We construct a simple model of education and growth in which children spend a fraction of their time and parents spend a fraction of their income on education. Both a strategic complementarity and an intergenerational externality are present. The interactions between each pair of consecutive generations lead to rich dynamics. We show that multiple growth equilibria arise, some of them periodic and some aperiodic. We also find a negative correlation between volatility and growth, without a one-way causal relationship between the two being, necessarily, present. Rather this negative correlation is driven by the structural characteristics of the economy.

Keywords: Education, Economic Growth, Business Cycle Volatility, Complex Dynamics

JEL Classification: E32; O41;

1 Introduction

The understanding of the fundamental causes that generate fluctuations in economic activity ('business cycles') has always occupied a prominent place in the research agenda of macroeconomists. According to the Keynesian tradition, the impulse source of business cycles comes from exogenous variations in aggregate demand conditions, which are propagated through the sluggish adjustment of prices; see, for example, Hicks (1937). For the proponents of the 'real business cycle' approach, price rigidity is not necessary as a propagation mechanism. Instead, the optimal response of agents to the presence of exogenous fundamental shocks and the process of capital accumulation suffice for these shocks to generate fluctuations in major macroeconomic variables, e.g., Long and Plosser (1983). A third strand of the literature went one step further by arguing that even fundamental shocks are not necessary for the emergence of cyclical movements in economic activity. Instead, non-monotonicities in the dynamic behavior of economic variables can generate periodic as well as aperiodic, but deterministic, orbits that resemble random ones, e.g., Grandmont (1985).

Another issue that was given serious consideration by researchers was the possibility that economic fluctuations may have deeper and permanent effects on long-term measures of macroeconomic performance. Although some early economists gave considerable thought to the issue, e.g., Schumpeter (1934), Kaldor (1954) and Kalecki (1954), this line of thought was brought to a standstill with the onset and the ensuing popularity of the Keynesian framework of business cycle analysis – a framework which, by its very nature, is focused on short-run issues and is not concerned with longer-term questions such as the fundamental determinants of economic growth. It would actually take many years before the interest in the links between growth and cyclical volatility was revived. This occurred with the emergence of the 'real business cycle' paradigm, which methodologically employed the neoclassical growth model so as to analyze the propagation of temporary fundamental shocks into fluctuations in macroeconomic variables. From that moment, it did not take long for researchers to realize that stochastic endogenous growth models could provide the analytical tools through which one can illustrate the correlation between aggregate variability and the long-run trend of output growth. These ideas were spurred even further by the appearance of many empirical papers which showed a statistically significant relationship between economic growth and measures of aggregate variability;

see, among others, Grier and Tullock (1989), Ramey and Ramey, (1995) and Fatas (2002). This literature has proposed a clear causality running from volatility to output growth, e.g., Femminis (2001), Canton (2002) and Blackburn and Varvarigos (2008). The basic idea is that macroeconomic volatility impinges on the individuals' behavior and actions concerning variables that determine the accumulation of growth promoting factors, such as physical and human capital accumulation.

Nonetheless, given that some of the reasons for the emergence of cyclical fluctuations are not necessarily exogenous, there is no a priori reason to presume that the empirically observed correlation between growth and volatility implies a causal link as well.[1] Addressing this argument successfully requires a model in which both the mechanisms that drive growth in the long-run and the incidence of growth volatility are attributed to the deep structural characteristics of the economic environment – i.e., they are endogenous. Whilst some authors have delved upon the construction and analysis of such frameworks, see, for example, Matsuyama (1999) and Wälde (2005), they focused purely on the conditions under which output growth may display periodic behavior over time.[2] Thus, it appears that there is still a void in the macroeconomics literature, resulting from the absence of a model that, in addition to reproducing endogenous growth cycles, will somehow correlate the incidence and/or magnitude of these cycles with the rate of economic growth, within the same framework of analysis.

Our paper aims at filling this void. It develops an overlapping generations model in which human capital constitutes the engine of growth.[3] The knowledge, skills and experience accumulated in the society are passed on to the next generation either through formal education and training or as the outcome of an "atmospheric" externality (intergenerational spillovers). While the latter form of knowledge transmission occurs freely, as one generation observes and imitates the other, the former requires the allocation of resources from the parents and the exertion of effort from the children. Moreover, these two "inputs" are complementary in the learning process, in the sense that an increase in one raises the marginal product of the other. The strategic interactions between generations and the presence of the aforementioned forms of knowledge transmission generate

[1]Indeed, Temple (1999) states that "there appears to be a negative relationship between output volatility and growth... as yet, the interpretation of the findings is unclear" (p. 145).

[2]For an example of a model that generates periodic equilibria in the real growth rate that are preference and not technology driven see the recent paper by Barnett and Bhattacharya (2008).

[3]Benhabib and Spiegel (1994) and Barro (2001), among many others, provide evidence regarding the contribution of human capital to growth and development.

rich dynamics. We show that multiple growth equilibria arise, some of them periodic and some aperiodic (chaotic). Moreover, the simultaneous existence of cyclical and stationary equilibria allow us to infer a negative correlation between volatility and growth. However, the novelty of our approach is that there does not exist a one-way causal link between the incidence of volatility and lower growth. Rather, both volatility and growth can be attributed to the intrinsic uncertainty of the economy. This is because, in the presence of permanent cycles, the growth rate is strictly lower than the one obtained under a stationary equilibrium. Finally, we demonstrate that if the mechanisms through which decisions are made change, then the endogenous fluctuations disappear and instead a unique balanced growth equilibrium emerges.

The remaining of the paper is organized as follows. Section 2 develops a rather standard model of human capital accumulation and growth. Section 3 derives the properties of the equilibrium and presents the implications for output growth. Section 4 considers an alternative form of interaction between generations, which removes the source of fluctuations and multiplicity. Section 5 concludes the main text, while an Appendix derives some auxiliary results.

2 A Parsimonious Model of Private Education

2.1 Description

We consider an overlapping generations economy in which time is discrete and indexed by $t \in \mathbb{N}$. Each period a new cohort of unit mass is born. Agents within each cohort are identical and live for two periods.[4] In the first period of their life they are "young adults" (or "offspring") and in the second "old adults" or ("parents"). All agents are endowed with one unit of time in each period. The young can allocate their time between activities that augment their human capital (e.g., formal schooling) and leisure. The old, on the other hand, supply their time, combined with their human capital (determining knowledge, efficiency and expertise), inelastically and receive the competitive wage per unit of human capital.

There is a single good in the economy. It is produced and supplied by perfectly competitive firms who employ effective labor so as to produce Y_t units of output according

[4] In an earlier version of the paper we assumed that agents live for an additional third period in which they are retired. Nevertheless, such an extension did not offer any additional insights and for the sake of expository simplicity we decided to omit it.

to

$$Y_t = AH_t, \qquad A > 0, \tag{1}$$

where H_t is the aggregate (and average) stock of human capital. H_t corresponds also to the economy's available units of effective labor because agents (whose large population is normalized to one) supply one unit of "raw" time each. Profit maximization and perfect competition imply that the equilibrium market wage per unit of (effective) labor is constant at $w_t = A \; \forall t$.

An agent born in period t enjoys utility over her entire lifetime according to the following simple functional form

$$u_t = \ln(1 - e_t) + \ln(c_{t+1}) + \ln(w_{t+2}h_{t+2}), \tag{2}$$

where e_t denotes schooling time when young, c_{t+1} stands for consumption when old and h_{t+2} is the human capital of an individual born in period $t+1$, which will be put to use in period $t+2$. A few remarks regarding the utility function (2) are in order. First, for the sake of simplicity, we assume that children do not consume or that the consumption of the children is included in the consumption of the parents. Second, the last term of the utility function indicates that parents are imperfectly altruistic towards their offspring. Specifically, a parent gets satisfaction by observing her offspring's income. This is meant to capture the idea that parents care about their offspring's future prospects and social status (both being enhanced through more advanced knowledge and/or increased income).[5] Finally, we remark that variations of this utility function abound in the literature; see, for example, Glomm (1997) and Ceroni (2001).

We assume that when young, a person can pick up a fraction $v \in [0, 1]$ of the existing (average) level of human capital H_t, without effort, simply by observing what the previous generation does.[6] The enhancement of an agent's human capital even further is possible only with the use of resources and time. Young adults must spend part of their time endowment on school in order to raise their level of human capital. Old adults (parents), on the other hand, spend resources (q_t) on their children's education. In other words, in each period a daughter devotes e_t units of her time and the mother allocates q_t units of her income towards the child's education. The two decisions are complementary, connecting

[5] Note that the inclusion of just the human capital of the offspring, h_{t+2}, instead of the income, $w_{t+2}h_{t+2}$, in the parental utility function (2) will not alter the results at all.

[6] The term $1 - v$ can be taken to capture the depreciation rate of the stock of knowledge.

thus generations within dynasties. We capture this simple fact of life with the following functional form

$$h_{t+1} = vH_t + \phi e_t q_t, \qquad \phi > 0, \tag{3}$$

where, it may be recalled that, h_{t+1} denotes the human capital of an agent born in period t. The specific functional form used in equation (3) allows us to derive analytical solutions for the decision rules regarding e_t and q_t. This human capital accumulation process shares common features with several papers in the literature; see, among others, De Gregorio and Kim (2000) and Ceroni (2001).

In the second period of life, parents allocate their income $w_{t+1}h_{t+1}$ between consumption c_{t+1} and their children's education q_{t+1}. Thus, an agent born in period t faces the following budget constraint in period $t+1$

$$c_{t+1} + q_{t+1} = w_{t+1}h_{t+1}. \tag{4}$$

2.2 Agents' problem

Agents born in period t choose $\{e_t, h_{t+1}, c_{t+1}, q_{t+1}\}$, in order to maximize their lifetime utility (given by equation 2) subject to the learning technology (3), the budget constraint (4) and the non-negativity constraints for e_t, c_{t+1}, and q_{t+1}, taking H_t, H_{t+1}, q_t and e_{t+1} as given. If e_t, c_{t+1}, and q_{t+1} are all positive then the first-order necessary conditions for this problem simplify as follows

$$\frac{1}{1-e_t} = w_{t+1}\phi q_t \frac{1}{c_{t+1}}, \tag{5}$$

$$\frac{\phi e_{t+1}}{vH_{t+1} + \phi e_{t+1}q_{t+1}} = \frac{1}{c_{t+1}}, \tag{6}$$

and the two constraints (3) and (4). Equations (5) and (6) describe, respectively, the optimal decision of an agent as a child (leisure vs. schooling) and as a parent (spending on child's education vs. spending on own consumption). They require that, at the optimum, the marginal benefit of each of these decisions be equal to the marginal cost.

We can view the situation described here as a game played between each pair of consecutive generations, that is, between parents and children. Each agent plays this game twice, once as a child and once as a parent. In any period $t+1$, a parent, who was born in period t, chooses the amount of resources q_{t+1} that she will allocate on her child's education. At the same time, the child chooses the amount of time, or equivalently the effort, e_{t+1} that she will devote on schooling. Nevertheless, when deciding her choice variable, each player takes the action of the other player as given.

If we combine (4) and (6), taking into account the non-negativity constraint $q_{t+1} \geq 0$, we obtain

$$q_{t+1} = \max\left\{0, \frac{w_{t+1}h_{t+1}}{2} - \frac{vH_{t+1}}{2\phi e_{t+1}}\right\} \qquad \forall t. \tag{7}$$

Obviously,

$$q_{t+1} = \begin{cases} >0 \quad \text{if} \quad e_{t+1} > \frac{vH_{t+1}}{\phi w_{t+1}h_{t+1}} \\ =0 \quad \text{if} \quad \quad otherwise \end{cases}. \tag{8}$$

Equation (7) constitutes the mother's reaction function. As indicated in (8), parents require that children put in a minimum level of schooling time/effort, before they decide to spend any resources on their education. Moreover, ceteris paribus, parents will not allocate any resources towards their children's education if their wage income $(w_{t+1}h_{t+1})$ or the efficiency of the education system (ϕ) are very small or if the rate at which human capital is transferred freely to the next generation (v) is very high. Furthermore, according to (7), even if positive, the exact income that parents spend on their children's education naturally depends positively on $w_{t+1}h_{t+1}$, ϕ and e_{t+1} and negatively on v. Finally, as it can be seen $dq_{t+1}/de_{t+1} > 0$, meaning that the daughter's schooling time is a (strategic) complement for the mother, that is, an increase in e_{t+1} raises the marginal return of education expenditure for the mother and makes her willing to increase her activity level.

Using (4) and (5), both updated one period, we can write

$$e_{t+1} = 1 - \frac{w_{t+2}h_{t+2} - q_{t+2}}{\phi w_{t+2}q_{t+1}},$$

or

$$e_{t+1} = \frac{1}{2}\left\{1 + \frac{q_{t+2} - w_{t+2}vH_{t+1}}{w_{t+2}\phi q_{t+1}}\right\} \tag{9}$$

which is the reaction function of the children. Thus, it is not clear whether q_{t+1} constitutes a (strategic) complement or substitute for the children, since the sign of de_{t+1}/dq_{t+1} depends on q_{t+2}, which depends on e_{t+2}, which depends on e_{t+3} and so on ad infinitum.

3 Equilibrium

A Nash equilibrium in this economy consists of sequences $\{e_t\}_{t=0}^{\infty}$, $\{c_t\}_{t=0}^{\infty}$, $\{q_t\}_{t=0}^{\infty}$, $\{Y_{t+1}\}_{t=0}^{\infty}$, $\{h_{t+1}\}_{t=0}^{\infty}$, $\{H_{t+1}\}_{t=0}^{\infty}$ and $\{w_t\}_{t=0}^{\infty}$, such that, given $h_0 > 0$, in every period

$t = 0, 1, \ldots,$ a) young agents choose e_t so as to maximize their utility function, taking q_t as given b) old agents choose q_t and c_t so as to maximize their utility, taking e_t as given c) firms choose effective labor so as to maximize their profits d) the labor market clears e) the goods market clears and f) $h_t = H_t$.

In equilibrium, since all young agents are identical $H_t = h_t \ \forall t$. Substituting this equilibrium condition in (7), we have that

$$q_{t+1} = \max\left\{0, \frac{vh_{t+1}}{2\phi}\left[\frac{1}{\gamma} - \frac{1}{e_{t+1}}\right]\right\} \qquad \forall t, \tag{10}$$

where $\gamma = v/A\phi$. Obviously,

$$q_{t+1} = \begin{cases} > 0 & \text{if} \quad e_{t+1} > \gamma \\ = 0 & \text{if} \quad \textit{otherwise} \end{cases}. \tag{11}$$

Also, if we substitute the equilibrium conditions $w_t = A$ and $H_t = h_t \ \forall t$ in (9), we obtain

$$e_{t+1} = \frac{1}{2}\left\{1 + \frac{q_{t+2} - Avh_{t+1}}{A\phi q_{t+1}}\right\}. \tag{12}$$

Next, substituting (10) in (12) and backdating one period, we have

$$e_{t+1} = f(e_t) = \frac{\gamma e_t(\gamma + e_t)}{-3e_t^2 + (2+\gamma)e_t - 2\gamma}, \tag{13}$$

which is a first-order non-linear ordinary difference equation in e_t.[7] This equation determines the dynamic path of the schooling time. Having determined this, simple substitution in (3) and (10) yields the equilibrium path of human capital and education expenditure. Similarly, the equilibrium paths of the remaining variables result from substitution in the relevant equations.

3.1 Stationary Equilibria

The steady-state values of (13) are

$$e_1^* = \frac{1}{3}\left(1 - \sqrt{1 - 3\gamma(2+\gamma)}\right) \quad \text{and} \quad e_2^* = \frac{1}{3}\left(1 + \sqrt{1 - 3\gamma(2+\gamma)}\right). \tag{14}$$

[7] We have also worked the solution with a utility function whose arguments are weighted differently and found that the qualitative characteristics of the equilibrium remain the same. We chose the present formulation, because it allows us to focus our attention on the parameters that are, indeed, crucial for the characterization of the equilibrium. These parameters are the ones included in the composite term γ.

As shown in the Appendix, the parameter γ must satisfy the following constraint

$$\frac{2}{\sqrt{3}} - 1 > \gamma > 0. \tag{15}$$

Intuitively, if γ is sufficiently high, which occurs for sufficiently high values of v (the rate at which human capital is transmitted freely to the next generation) or for sufficiently low values of A or ϕ (the efficiency parameter in production or in the education system, respectively), then no resources or time will be devoted to education.[8]

The graph of (13) approaches asymptotically the values $\underline{e}$ and $\overline{e}$ (defined in the Appendix), where $1 > \overline{e} > \underline{e} > 0$. Under the restriction on γ given in (15), it follows from (14) that $1 > e_2^* > e_1^* > 0$. There are two main configurations shown in Figures 1a and 1b.[9] Let the equilibria e_1^* and e_2^*, not shown in the graphs to reduce clutter, correspond to the lower and higher intersection, respectively, of the graph of f with the 45-degree line. Then Figures 1a and 1b are drawn so that $e_1^* < e_{\min}$ in the former and $e_1^* > e_{\min}$ in the latter.

The reason why the relationship between e_t and e_{t+1} is non-monotonic is because there are two effects present. On the one hand, if parents expect an increase in e_{t+1} they will raise q_{t+1}, which will decrease their consumption and their leisure, i.e., will raise e_t. On the other hand, each unit of time invested in education yields now a higher rate of return (this is captured by the term $w_{t+1}\phi q_t$ in equation 5); this generates an income effect, which will induce parents to raise their leisure and offset the initial increase in e_t (with the higher rate of return to education, parents can now earn the same income with a lower effort e_t). The interaction of these two effects gives rise to the U shaped curve of e_{t+1} as a function of e_t drawn in Figures 1a and 1b.

Next we examine the stability of the two equilibria. The higher equilibrium is always repelling since $f'(e_2^*) > 1$. The stability property of the lower equilibrium e_1^*, however, depends on the value of γ. For values of $\gamma \in (0, 0.147)$, $f'(e_1^*) < -1$ and hence e_1^* is a repelling fixed point; any orbit that starts near e_1^* exhibits explosive oscillations (this

[8] As shown in the Appendix (see equation R4), $\gamma \in [0, (2/\sqrt{3}) - 1]$. We consider the case where $\gamma = (2/\sqrt{3}) - 1$ in footnotes 9 and 10 below and the case where $\gamma = 0$ at the end of Subsection 3.2 (see Proposition 3).

[9] There are three more possible configurations, which are not shown here. The first is the case where $e_1^* = e_{\min}$. This occurs when γ takes the value γ' (defined in the Appendix). The second configuration that is not depicted is the case where $e_1^* = e_2^*$. This occurs when γ takes the value $(2/\sqrt{3}) - 1$. In this case the graph of the difference equation is tangent to the 45^0 line. The last configuration that is not shown is the one where the graph of f does not intersect the 45^0 line. This occurs when γ does not satisfy (15), that is $\gamma > (2/\sqrt{3}) - 1$. No interior equilibrium exists in this case.

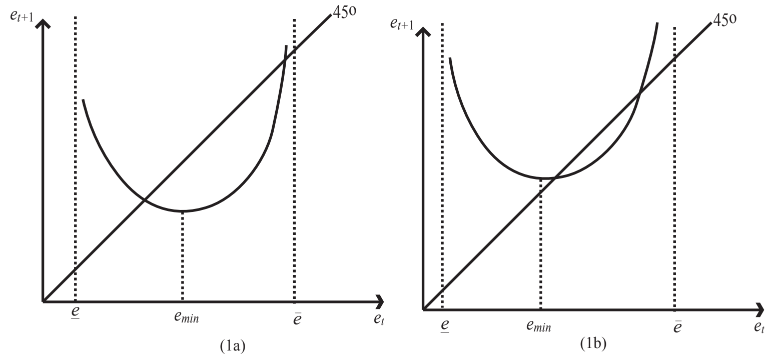

Figure 1. Steady-state Equilibria

occurs in the case depicted in Figure 1a, when e_1^* is sufficiently far from $e_{\min}$). For values of $\gamma \in (0.147, \gamma')$, where $\gamma' \simeq 0.153$ (see the Appendix), $-1 < f'(e_1^*) < 0$; nearby orbits converge to e_1^* through damped oscillations (this occurs in the case depicted in Figure 1a, when e_1^* is sufficiently close to $e_{\min}$). Finally, if $\gamma \in (\gamma', (2/\sqrt{3}) - 1)$, then $0 < f'(e_1^*) < 1$ and the convergence to e_1^* is monotonic (this occurs in the case depicted in Figure 1b).[10] We summarize our results so far as follows.

Proposition 1. *Equilibria are defined for values of $\gamma \in [0, (2/\sqrt{3}) - 1]$. If γ lies in the interior of this interval, then there exist two steady-state equilibria, $e_2^* > e_1^*$. The highest equilibrium e_2^* is always repelling, whereas the lowest equilibrium e_1^* is either repelling or attracting depending on the value of γ, which is a composite term consisting of structural parameters of the economy.*

3.2 Dynamic Equilibria

The difference equation (13) accepts also periodic equilibria. Indeed, consider the second iterate of f, $f^2(e_t) = f(f(e_t))$. This map has four fixed points, e_1^*, e_2^*, given in (14), and

$$e_{c1}^*, e_{c2}^* = \frac{-\gamma^2 + \gamma + 2 \pm \sqrt{\gamma^4 + 2\gamma^3 + 5\gamma^2 - 28\gamma + 4}}{2(\gamma + 4)}. \tag{16}$$

The last two fixed points of f^2, e_{c1}^*, e_{c2}^*, correspond to a 2-period cycle. Note that these are real and distinct if the quantity under the square root in (16) is positive, which requires that $\gamma \in (0, 0.147)$. Also, over this range of γ both of these points lie between $\underline{e}$ and $\bar{e}$. To analyze the stability of the cycle we have to compute its multiplier $(f^2)'(e_c^*) = f'(e_{c1}^*) f'(e_{c2}^*)$. Straightforward computation shows that the absolute value of

[10] If $\gamma = 0.147$, then $|f'(e_1^*)| = 1$ and e_1^* is a neutral fixed point Also, at the value $\gamma = (2/\sqrt{3}) - 1$, the family of functions f_γ undergoes a saddle-node bifurcation; see Devaney (2003, p. 88). The number of fixed points is two for $\gamma < (2/\sqrt{3}) - 1$, one for $\gamma = (2/\sqrt{3}) - 1$, and zero for $\gamma > (2/\sqrt{3}) - 1$.

the multiplier is less than unity, and hence the 2-period cycle is stable, for parameter values $\gamma \in (0.143, 0.147)$. Recall from the previous subsection that e_1^* is an attracting (repelling) point if γ is higher (lower) than 0.147. Thus, as γ crosses 0.147, decreasing from $(2/\sqrt{3}) - 1$ towards 0, the fixed point e_1^* found above changes from attracting to repelling and, at the same time, gives birth to an attracting 2-period cycle (the emergence of a 2-period cycle is illustrated in Figure 2).

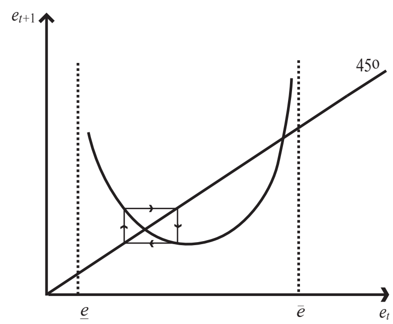

Figure 2. A 2-period Cycle

A 2-period cycle, however, is not the only periodic equilibrium. In fact as γ crosses the value 0.143, decreasing towards 0, the 2-period cycle changes from attracting to repelling and, at the same time, gives birth to an attracting 4-period cycle. This process continues; that is, there is a decreasing sequence of bifurcation points $\{(2/\sqrt{3}) - 1, 0.147, 0.143, \ldots\}$, such that for values of γ between any two consecutive members of the sequence γ_k and γ_{k+1} the prime 2^k-period solution is stable, while the periodic solutions of all other periods $2, 4, \ldots, 2^{k-1}$ become unstable. The last period that can arise in this bifurcation process is period three and consequently by Sarkovskii's Theorem there are periodic points of all other periods (Figure 3 shows an example of a 3-period cycle). This phenomenon is known as *the period-doubling route to chaos,* see, for example Devaney (2003, p.130).

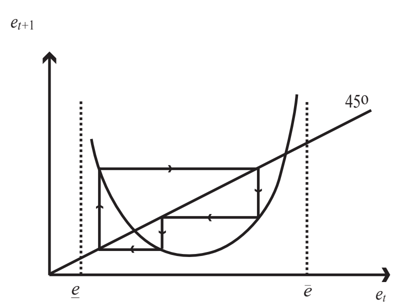

Figure 3. A 3-period Cycle

A concise way to illustrate the aforementioned qualitative changes in the nature of the smaller equilibrium is the bifurcation diagram for equation (13), which is presented in Figures 4a and 4b. On the vertical axis we measure the orbit of a point near e_1^* (we choose as initial condition $e_0 = 0.25$) and on the horizontal axis the parameter γ (Figure 4a covers the entire range of γ, while Figure 4b represents a magnification of it covering a smaller range of γ). These diagrams are best understood if we start from the value of $\gamma = (2/\sqrt{3}) - 1$, which is the maximum permissible value of γ, and move towards the origin. Initially, there exist two fixed points e_1^* and e_2^*; the first is stable and the second unstable. As γ decreases the nature of the first one goes through some sudden changes described above (the properties of e_2^* do not change and hence we are not concerned with it). Thus, for $\gamma = 0.147$ there is a 2-period cycle, whereas for $\gamma = 0.143$ a 4-period cycle arises. This period-doubling process continues as γ decreases. Eventually, this process stops around $\gamma = 0.14$. Indeed, notice that for lower values of γ no attracting solution exists.[11]

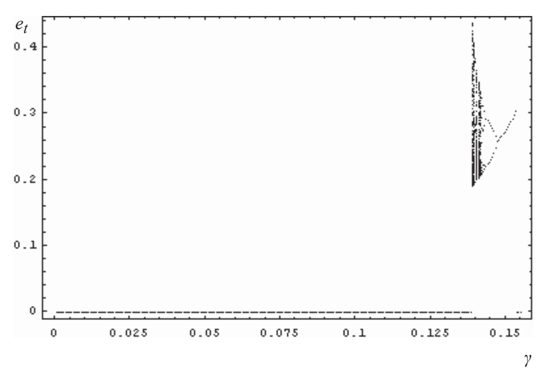

Figure 4a. Bifurcation Diagram

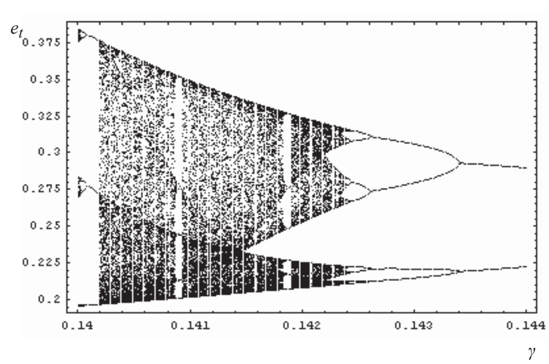

Figure 4b. A Magnification of the Bifurcation Diagram

[11]Notice that, in Figure 4a, the long-run value for e_t appears to be zero for a wide range of permissible values of γ. This is because the software we used to generate this graph cannot take account of the fact that the initial condition of e_t is not pinned down by the model. What actually happens is that, for these values of γ, agents will choose directly either e_1^* or e_2^*.

A well-known characteristic of chaotic dynamical systems is the exponential divergence of orbits starting arbitrarily close to each other. This is shown by computing the Liapunov number of an orbit, see, for example, Kulenović and Merino (2002).[12] Figure 5 shows the Liapunov number of an orbit staring at $e_0 = 0.25$ when $\gamma = 0.1402$. The graph indicates that the orbit is chaotic since its Liapunov number is clearly greater than unity.

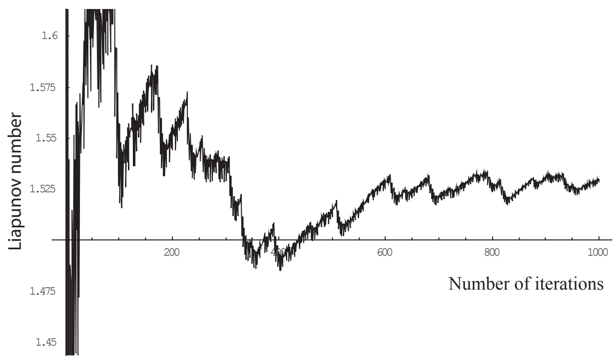

Figure 5. Liapunov Number

After determining the mathematical properties of the solutions over the entire range of permissible values of γ, let us return to the equilibrium path that will be selected by forward looking agents. As indicated above the interval $(0.14, (2/\sqrt{3}) - 1)$ comprises smaller intervals (γ_k, γ_{k+1}), where each γ_k is a bifurcation value. Within each interval (γ_k, γ_{k+1}), there is at most one attracting periodic solution; this result is known as Singer's Theorem, see Kulenović and Merino (2002, p. 21).[13] For parameter values $\gamma \in (0, 0.14)$ there are no periodic solutions, while both fixed points e_1^* and e_2^* are unstable. Since e_t is not a predetermined variable and hence there does not exist any initial condition for it, it follows that for any value of γ below approximately 0.14 there exist two possible equilibria, e_1^* and e_2^*, whereas for values of γ greater than 0.14 there exist infinitely many equilibria, some of them periodic and some aperiodic).

Finally, with regard to the equilibrium path of the other variables, it is easy to show that over the range of γ given by (15), $\underline{e} > \gamma$ and hence $e_{t+1} > \gamma$; consequently $q_{t+1} > 0$ (see equation 11). Given then the equilibrium path of e_{t+1}, the behavior of q_{t+1}, h_{t+1} and c_{t+1} follows from the system of equations (3), (5) and (10).[14]

[12] The Liapunov number L is defined as $L(x_0) = e^{\lambda(x_0)}$, where λ is the Liapunov exponent. "Intuitively, the Lyapounov exponent measures the average (exponential) rate of separation of the orbits of points near x from the trajectory generated by x," Grandmont (2008, p. 163). In particular, an orbit is chaotic if the Liapunov exponent is positive or, equivalently, the Liapunov number is greater than unity.

[13] The sequence $\{\gamma_k\}_{k=0}^{\infty}$ has the property that $\lim_{k\to\infty}[(\gamma_k - \gamma_{k-1})/(\gamma_{k+1} - \gamma_k)$ =Feigenbaum's number $\simeq 4.66920$, see, for example, Kulenović and Merino (2002 p. 21).

[14] In Palivos and Varvarigos (2009) we analyze a model in which education is publicly provided, while

The following proposition summarizes our results so far in this subsection.

Proposition 2. *If γ takes values in the interval $(0, 0.14)$ then there are are two unstable equilibria; which one will be selected depends on each generation's beliefs about the behavior of the other generation. For values of γ in the interval $(0.14, (2/\sqrt{3}) - 1))$ there are infinitely many equilibria, some of them periodic and some aperiodic (chaotic).*

The study of $\gamma = 0$ deserves particular attention. Recall that $\gamma = 0$ is implied by $v = 0$, that is, there is no externality in the transmission of knowledge and children learn only through formal education. Substituting $\gamma = 0$ in (10) we get $q_{t+1} = Ah_{t+1}/2$. Also, equations (4) and (5) yield $c_{t+1} = Ah_{t+1}/2$ and $e_{t+1} = 2/3$. Finally, substituting in the learning technology (3) we have $h_{t+2} = \phi Ah_{t+1}/3$. Thus,

Proposition 3. *If $\gamma = 0$ then there is a unique balanced growth equilibrium path, and there no transitional dynamics, i.e., $e_t = 2/3$, while all other variables grow at a constant rate $\forall t$.*

As it can be seen from (7), the reason behind the uniqueness of the equilibrium is that with $\gamma = 0$, the children's schooling time is not a strategic complement for the parents any more. Instead, the parents spend a fixed share of their income on their children's education.[15]

3.3 Implications for Output Growth

Given $h_t = H_t$ and equations (1), (3) and (10), we can write the (gross) growth rate of output as

$$\frac{Y_{t+1}}{Y_t} = \frac{v}{2}\left(1 + \frac{e_t}{\gamma}\right) \equiv G(e_t). \tag{17}$$

Evidently, the stationary and dynamic properties of the growth rate are fully described by the corresponding properties of e_t. Thus, the possibility of periodic behavior for e_t implies the emergence of endogenous growth cycles. Furthermore, such equilibria arise in the neighborhood of e_1^*. Combining this with $e_1^* < e_2^*$ and $G' > 0$, our model surmises a negative correlation between volatility and growth.

Proposition 4. *Economies that select equilibria in the neighborhood of e_1^* will exhibit lower growth rate and, possibly, higher volatility compared with economies that select the*

parents vote on the tax rate that finances education spending. Interestingly, the complicated dynamics of e_t do not emerge in that setting because individuals do not internalize the effect of their own schooling decisions on aggregate income, which partially determines the level of public spending on education.

[15]These results follow from the particular preferences that we use here.

high action equilibrium e_2^*.

The idea that there is a negative correlation between volatility and growth is supported by strong empirical evidence; see, for example, Ramey and Ramey (1995). However, our framework suggests a novel explanation for its emergence; mainly, that it is not greater volatility that necessarily causes a fall in the rate of economic growth. Rather, it is the situation leading to relatively lower growth that it is also responsible for generating permanent endogenous cycles. Hence, the link between volatility and growth is embedded to the structural characteristics of the economic environment and there is no a priori reason to presume any clear causality between the two phenomena. Volatility may be as much a cause of lower growth as lower growth is a cause of greater volatility. We illustrate this by showing that if the mechanisms through which agents make decisions change, then the endogenous fluctuations and the multiplicity of equilibria disappear.

4 Sequential Choices

In the model analyzed so far there is no structural uncertainty, but there is strategic uncertainty; that is, there is uncertainty regarding the strategy of the "other player". This uncertainty is among the structural characteristics of the economy, as reflected in its institutions and the mechanisms through which decisions are made. In this section we alter the model so as to eliminate strategic uncertainty and show that doing so eliminates endogenous fluctuations; instead, a unique balanced growth equilibrium emerges.

Consider the case where children commit to a certain level of schooling time/effort and, given this choice, parents decide on the level of resources spent on education.[16] In terms of a concrete real-life example, we may think of scholarships and/or tuition fee waivers that are provided on the basis of students' success on achieving some performance targets. In other words, as before parents who belong to generation t maximize (2) subject to (3) and (4). The solution to this problem leads to (7). Knowing (7), children who belong to generation $t+1$ maximize (2) subject to (3), (4) (all updated one period) and (7). This maximization leads to

$$\frac{1}{1-e_{t+1}} = w_{t+2}\phi\frac{1}{c_{t+1}}\left(q_{t+1} + e_{t+1}\frac{\partial q_{t+1}}{\partial e_{t+1}}\right), \tag{18}$$

[16] The case where parents commit is particularly complicated, because as mentioned before the children's reaction function includes the variable q_{t+2}, that is, the amount of resources that they intend to spend on their children's education (see equation 12). Since this equation is a constraint to the parent's problem, we end up with a dynamic system of difference equations, one of which is of degree higher than one.

and

$$\frac{\phi e_{t+2}}{vH_{t+2}+\phi e_{t+2}q_{t+2}}=\frac{1}{c_{t+2}}. \tag{19}$$

The definition of the equilibrium is the same as in the previous case with the exception of condition a), which now becomes a$'$) the young agents choose e_t so as to maximize their utility function, taking into account that the amount of resources that their parents will allocate on their education is related to their education effort, i.e., q_t is a function of e_t and is given by equation (10).

After applying the equilibrium condition $H_t = h_t \ \forall t$, some straightforward but tedious algebra produces

$$e_{t+1}=g(e_t)=\frac{\gamma(\gamma+e_t)}{-3e_t+2-\gamma}. \tag{20}$$

Equation (20) is the counterpart of (13). For a positive value of e_{t+1}, it is required that $e_t<(2-\gamma)/3$. Moreover, recall from (10) that for a positive value of q_{t+1}, it must be the case that $e_{t+1}>\gamma$ or, using (20), $e_t>(1-\gamma)/2$. It is easy to show that $g'(e_t)>0$ and $g''(e_t)>0$. The reason behind the monotonicity of the $g(e_t)$ function is that the rate of return to education, which in any period $t+1$ is now equal to $w_{t+1}(\partial h_{t+1}/\partial e_t)=w_{t+1}\phi(q_t+e_t(\partial q_t/\partial e_t))=w_{t+1}h_t/2$, i.e., the term inside the parenthesis in (18) is independent of q_t. On the contrary, in the previous case the rate of return to education was equal to $w_{t+1}\phi q_t$ (compare equations 5 and 18). Thus, the income effect that generated the downward sloping part of (13) does not exist anymore.

The steady-state values of (20) are

$$e_1'=\frac{1}{3}\left(1-\gamma-\sqrt{-2\gamma^2-2\gamma+1}\right) \quad \text{and} \quad e_2'=\frac{1}{3}\left(1-\gamma+\sqrt{-2\gamma^2-2\gamma+1}\right). \tag{21}$$

Note that $e_1'<(1-\gamma)/2$ for every positive value of γ and hence it is rejected. On the other hand, $(1-\gamma)/2<e_2'>(2-\gamma)/3$, if $0<\gamma<1/3$.[17] The phase diagram of (20) is depicted in Figure 6. Given that e_2' is a repelling point and that e_t is not a predetermined variable, there is only one equilibrium $\{e_t\}=\{e_2'\}\ \forall t$. In other words, there is no source of endogenous fluctuations or multiplicity left in the model.[18] The behavior of the other variables follows then easily from the relevant equations.

[17] Within this range of γ, $-2\gamma^2-2\gamma+1>0$ and hence e_2' is real and distinct from e_1'.

[18] For a more direct investigation of the relation between strategic complementarity and the multiplicity of equilibria, see Cooper and John (1988).

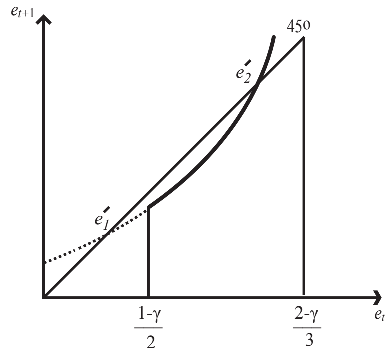

Figure 6. A Unique Steady-state Equilibrium

Proposition 5. *If decisions are taken sequentially and strategic uncertainty is resolved then the endogenous fluctuations and multiplicity disappear and a unique balanced growth equilibrium emerges.*

5 Conclusions

We have presented a simple model of education and growth, which shares a lot of common features with previous models in the literature. We have shown that the strategic interactions in the learning technology between every pair of consecutive generations, parents and children, and the presence of an intergenerational externality lead to multiple growth equilibria, some of them periodic and some aperiodic. Crucial determinants of the dynamic equilibrium are the rate at which human capital is transmitted to the next generation as well as the efficiency in the education and output sectors. The model also creates volatility of the growth rate that emanates endogenously from the fundamentals of the economy. Finally, we have shown that if we change the mechanisms through which decisions are made then the endogenous fluctuations and multiplicity disappear and a unique balanced growth equilibrium emerges.

Finally, we would like to conclude with the following comment, which offers a more general perspective. The paper analyzed the interactions in knowledge transmission between consecutive generations. Abstracting from the particular context, we believe that our framework captures more general features regarding intergenerational interactions that can lead to endogenous fluctuations. That is, similar games between generations abound in real life. For example, H could be a common resource, which is affected by the actions of overlapping generations. Hence, we think that other intertemporal issues can be analyzed along the lines presented in this paper.

Appendix

A.1 Restrictions on γ

For q_t to be positive, equation (10) requires that $e_{t+1} > \gamma$ and since $e_t < 1 \; \forall \; t$, we have that

$$\gamma < 1. \tag{R1}$$

Furthermore, it follows from (13) that $e(t+1)$ takes positive values if $-3e_t^2+(2+\gamma)e_t-2\gamma > 0$. This requires that $e(t)$ takes values within the interval $(\underline{e}, \overline{e})$, where

$$\underline{e}, \overline{e} = \frac{(2+\gamma) \pm \sqrt{\gamma^2 - 20\gamma + 4}}{6}. \tag{A1}$$

For $\underline{e}$ and $\overline{e}$ to be real numbers, it must be the case that $\gamma^2 - 20\gamma + 4 > 0$ or that

$$\gamma < 2(5 - 2\sqrt{6}) \text{ or } \gamma > 2(5 + 2\sqrt{6}). \tag{R2}$$

Finally, it follows from (14), that the steady-state values e_1^* and e_2^* exist if $1-3\gamma(2+\gamma) \geq 0$, which requires that

$$\frac{2}{\sqrt{3}} - 1 \geq \gamma \geq -\frac{2}{\sqrt{3}} - 1 \tag{R3}$$

Combining (R1), (R2) and (R3), we conclude that

$$\frac{2}{\sqrt{3}} - 1 \geq \gamma \geq 0. \tag{R4}$$

A.2 Properties of (13)

Let

$$e_{t+1} = f(e_t) = \frac{\gamma e_t(\gamma + e_t)}{-3e_t^2 + (2+\gamma)e_t - 2\gamma}.$$

Notice that $f(e_t)$ approaches infinity as e_t approaches $\underline{e}$ from above or $\overline{e}$ from below, where $\underline{e}$ and $\overline{e}$ are given in (A1) above. Also, straightforward differentiation yields

$$f'(e_t) = \frac{2\gamma\left[(2\gamma+1)e_t^2 - 2\gamma e_t - \gamma^2\right]}{(-3e_t^2 + (2+\gamma)e_t - 2\gamma)^2}, \; f''(e_t) = \frac{4\gamma\left[3(2\gamma+1)e_t^3 - 9\gamma e_t^2 - 9\gamma^2 e_t + 4\gamma^2 + \gamma^3\right]}{(-3e_t^2 + (2+\gamma)e_t - 2\gamma)^3}.$$

Note that $sign[f'(e_t)] = sign\left[(2\gamma+1)e_t^2 - 2\gamma e_t - \gamma^2\right]$. Given that e_t takes non-negative values, it follows that

$$e_t \lesseqqgtr e_{\min} \Rightarrow f'(e_t) \lesseqqgtr 0, \quad \text{where} \quad e_{\min} \equiv \frac{\gamma\left(1 + \sqrt{2(1+\gamma)}\right)}{1+2\gamma}. \tag{A2}$$

Next consider the $sign\,[f''(e_t)] = sign\,[3(2\gamma+1)e_t^3 - 9\gamma e_t^2 - 9\gamma^2 e_t + 4\gamma^2 + \gamma^3]$. As it can be easily shown, the sign of the minimum value of $f''(e_t)$, over the range of permissible values of γ given in (15), is positive, which implies that $f(e_t)$ is strictly convex. Moreover, using the above formulas, we can show that for values of γ given in (15), $1 > \overline{e} > e_2^* > e_1^* > \underline{e} > 0$. Finally,

$$\gamma \lesseqgtr \gamma' \Leftrightarrow e_1^* \lesseqgtr e_{\min}, \text{ where } \gamma' = -1 + \frac{1}{2}\frac{1}{3^2}\frac{(z^2+2z-14)^2}{z^2} \simeq 0.53, \;\; z = [2(31+3\sqrt{183})]^{1/3}.$$

The graph of $f(e_t)$ is then given in Figures 1a and 1b (see also footnote 9).

A.3 Derivation of (16)

Consider the map f given in equation (13). The second iterate of f, $f^2(e) = f(f(e))$ is

$$f^2(e) = \frac{\gamma^2(\gamma+e)(e^2-e-\gamma e+\gamma)}{(3\gamma+12)e^4 + (4\gamma^2-5\gamma-14)e^3 + (\gamma^3+16\gamma+4)e^2 + (\gamma^3-2\gamma^2-8\gamma)e + 4\gamma^2}.$$

A periodic point of period 2 is a fixed point of the f^2 map. Solving the equation $f^2(e) = e$, we find four such fixed points. Two of them are the fixed points of the map f, which is expected since fixed points of f are also fixed points of each iterate of f. The other two fixed points are the ones given in (16).

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