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# Do Inada conditions imply that production function must be asymptotically Cobb–Douglas? A comment

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**Abstract**

We correct an intermediate mistake in Barelli and Pessôa [Barelli P. and Pessôa S. de A., 2003, "Inada conditions imply that production function must be asymptotically Cobb–Douglas," Economics Letters 81, 361–363] and show that the main result is still valid. We also show that Barelli and Pessôa wrongly identified the class of functions with elasticity of substitution asymptotically equal to one as the Cobb–Douglas class.


*Keywords:* Inada conditions; Elasticity of substitution; Production function

*JEL classification:* E13; E23

## 1. Introduction

In a recent paper Barelli and Pessôa (2003) (henceforth BP) claimed that Inada conditions imply that a strictly increasing and strictly concave (per capita) production function $f(k)$ must be asymptotically Cobb–Douglas. What BP actually proved is that Inada conditions imply that the elasticity of substitution must be asymptotically equal to 1. That is,

$$f(0) = 0 \text{ and } f'(0) = \infty \Rightarrow \sigma(0) = 1. \tag{1}$$

(A similar result is shown for the case where $k \to \infty$).

In this note, we do two things: first we show that there is an intermediate mistake in BP. We correct it, and show that the main result in BP is still valid; second we provide an example of a function whose elasticity of substitution is asymptotically equal to one but it is not the Cobb–Douglas functional form. That is, we show that BP wrongly identified the class of functions with elasticity of substitution asymptotically equal to one as the Cobb–Douglas class.

## 2. Result

The proof of (1) is based on the following two intermediate results, which are also interesting in their own right (see Proposition 1 in BP):

$$\text{If } \sigma(0) < 1 \text{ then } f(0) = 0 \text{ and } f'(0) < \infty \tag{2a}$$

$$\text{If } \sigma(0) > 1 \text{ then } f(0) > 0 \text{ and } f'(0) = \infty. \tag{2b}$$

Indeed, combining the contrapositive statements of (2a) and (2b) yields (1). Nevertheless consider the following mixed CES-linear production function found in, among others, de la Croix and Michel (2002, p. 122): $f(k) = b + \frac{ak}{1+k}$, $a$, $b>0$. For this production function we have $\sigma(0)>1$, $f(0)=b>0$ but $f'(0)=a<\infty$. A second counter-example to (2b) is the following: $f(k)=Ak+\alpha-\beta\exp(-\beta k)$, $A>0$, $\alpha>\beta>0$, where $\sigma(0)>1$, $f(0)=\alpha-\beta>0$ but $f'(0)=A+\beta^2<\infty$. Finally, another way to present the problem is the following: combining the contrapositive statements of (2a) and (2b) yields, in addition to (1), $f(0)>0$ and $f'(0)<\infty \Rightarrow \sigma(0)=1$ which, as each of our two examples

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demonstrates, is false. Obviously, these results cast doubt to the validity of (1).

By analyzing carefully the proof of (1) in BP, we see that what they actually show is (see the last part of their proof of Proposition 1 on p. 363)

$$\text{If } \sigma(0) < 1 \text{ then } f(0) = 0 \text{ and } \lim_{k\to 0}\frac{f(k)}{k} < \infty \tag{2a$'$}$$

$$\text{If } \sigma(0) > 1 \text{ then } f(0) > 0 \text{ and } \lim_{k\to 0}\frac{f(k)}{k} = \infty. \tag{2b$'$}$$

Then by assuming implicitly that $\lim_{k\to 0}\frac{f(k)}{k} = f'(0)$, they arrive at (2a) and (2b) and hence at (1) (a similar argument is developed for the case where $k\to\infty$). Nevertheless, while $\lim_{k\to\infty}\frac{f(k)}{k} = f'(\infty)$ is always true, $\lim_{k\to 0}\frac{f(k)}{k} = f'(0)$ is not. Indeed, consider the following lemma.

**Lemma.** *If either $f'(0)=\infty$ or $f(0)=0$ then* $\lim_{k\to 0}\frac{f(k)}{k} = f'(0)$.

**Proof.** If $f'(0)=\infty$ and $f(0)\neq 0$ then the $\lim_{k\to 0}\frac{f(k)}{k} = \infty = f'(0)$ follows immediately. If $f(0)=0$ then the result follows from the definition of the derivative. □

In both counter-examples provided above $f(0)>0$ and $f'(0)<\infty$ hence $\lim_{k\to\infty}\frac{f(k)}{k} = \infty \neq f'(0) < \infty$ and thus (2b) is incorrect. Fortunately, however, (1) is still valid, since when the two Inada conditions are combined then $\lim_{k\to 0}\frac{f(k)}{k} = f'(0)$ thus, the transition from

$$f(0) = 0 \text{ and } \lim_{k\to 0}\frac{f(k)}{k} = \infty \Rightarrow \sigma(0) = 1, \tag{1$'$}$$

which follows from (2a′) and (2b′), to (1) is legitimate.[1]

## 3. Interpretation

Next we turn to the asymptotic behavior of two production functions that exhibit the same limiting value of the elasticity of substitution. More specifically, we ask the following question: Does a function that satisfies the Inada conditions, and hence has a limiting value of $\sigma$ equal to unity, behave asymptotically like the Cobb–Douglas?

First consider the following broadly accepted definition (see, for example, Zorich (2004)).

**Definition.** A function $f(x)$ is said to behave asymptotically as $x\to a$ like a function $g(x)$, if $\lim_{x\to a}\frac{f(x)}{g(x)} = 1$.

Next consider the following production function:

$$f(k) = A\left[(1+k^{-\rho})^{-\frac{1}{\rho}} - k - 1\right], \quad -1<\rho<0. \tag{3}$$

This production function satisfies all four Inada conditions: $f(0)=0$ and $f'(0)=\infty$, $f'(\infty)=0$ and $f'(\infty)=\infty$. Thus, as BP have proved, $\sigma(0)=1$ and $\sigma(\infty)=1$. Nevertheless, (3) does not behave asymptotically like the Cobb–Douglas. Indeed, straightforward calculations show that the limit of the ratio of (3) to the Cobb–Douglas function $Ak^{\alpha}$, $\alpha\in(0,1)$, as $k$ approaches infinity, takes values different from unity; for example, if $\rho=-0.5$ and $\alpha=0.5$ then, as $k$ approaches infinity, the ratio tends to 2, while if $\rho=-0.2$ and $\alpha=0.5$, then it tends to infinity. Hence, we conclude that a production function that satisfies the Inada conditions does not necessarily behave asymptotically like the Cobb–Douglas, even though the limiting value of its elasticity of substitution is equal to the value of the elasticity of substitution of the Cobb–Douglas production function, which is constant and equal to one.

In conclusion, BP conveys an important message, namely that when we assume that the Inada conditions hold we simultaneously assume that the limiting value of the elasticity of substitution is equal to unity, which is a rather restrictive assumption. Furthermore, BP does make the valid claim that if evidence points out that Cobb–Douglas are not appropriate, then Inada conditions cannot be imposed.[2]


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## References


Barelli, P., Pessôa, S. de A., 2003. Inada conditions imply that production function must be asymptotically Cobb–Douglas. Economics Letters 81, 361–363.

de la Croix, D., Michel, P., 2002. A Theory of Economic Growth. Cambridge University Press, Cambridge.

Zorich, V.A., 2004. Mathematical Analysis I. Springer-Verlag, Berlin.


[1] Note that other pair that results from the combination of the contrapositive statements of (2a′) and (2b′), namely, $f(0)>0$ and $\lim_{k\to 0}\frac{f(k)}{k} < \infty$, is incompatible.

[2] An anonymous referee pointed this out to us.